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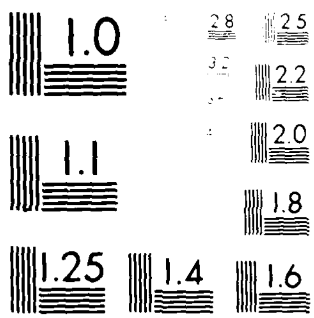
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MRC Technical Summary Report #2348

# STABILITY THEORY OF A CONFINED TOROIDAL PLASMA PART I. EXISTENCE AND UNIQUENESS

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# STABILITY THEORY OF A CONFINED TOROIDAL PLASMA
## PART I. EXISTENCE AND UNIQUENESS

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ABSTRACT

The linear MHD stability of a confined plasma is generally studied by means of energy principles. Up to date, these energy principles have never been justified rigorously, and the existence of a solution to the linearized equations is also tacitly assumed. In this report, based upon a variational approach, we shall first establish the existence and uniqueness of a generalized solution to the linearized Lundquist equations for a toroidal plasma confined in a conducting shell. In a subsequent report, the so-called modified energy principle, which includes linear energy principle as a special case, will be justified rigorously and a solid foundation is then laid for the application of these energy principles to the linear MHD stability of a confined plasma.

AMS(MOS) Subject Classification: 76W05, E25

Key Words: Lundquist equations, toroidal plasma, variational formulation, existence and uniqueness

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STABILITY THEORY OF A CONFINED TOROIDAL PLASMA
PART I. EXISTENCE AND UNIQUENESS

Peter Laurence* and M. C. Shen†

## §1. Introduction.

It is now well recognized that controlled thermonuclear fusion may offer the ultimate solution to the world energy crisis with inexhaustible resources for years to come. One of the most promising methods at present to achieve the goal of a self-sustaining thermonuclear reaction is the magnetic confinement of a plasma inside a toroidal vacuum chamber shielded by a conducting shell, such as a tokomak. General surveys about the progress of controlled thermonuclear research may be found in the recent articles by Grad (1977) and Furth (1980), among others. Needless to say, physical and mathematical problems abound in the study of a magnetically confined plasma, such as equilibrium, stability, wave motion, diffusion and adiabiatic compression, just to name a few. However, at present the problem of plasma stability still remains the central issue in fusion research, especially based upon magnetohydrodynamic (MHD) models.

Traditionally the standard procedure to study MHD stability of a plasma with simple geometry is to linearize the governing equations about a given equilibrium state and analyze the linearized equations via normal modes. Evidently this method is not very useful wl plicated geometries are considered. To avoid this difficulty, Berstei al. (1957) developed a

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variational approach to linear MHD stability, and formulated the so-called linear energy principle. Its advantage lies in the fact that to determine stability of a plasma it is necessary only to discover whether there is a perturbation which decreases the potential energy from its equilibrium state. Since then a vast amount of literature based upon this technique has appeared. In obtaining the necessary condition for stability, their arguments rely upon the assumption that the eigenfunctions of a certain linear operator must form a complete orthonormal basis. This weakness was then removed by Laval et al. (1965). In their derivation they introduced a modified energy principle for the so called $\sigma$-stability of a confined plasma, which includes the linear energy principle as a special case. Intuitively speaking, a plasma is called $\sigma$-stable if it does not grow faster than $\exp(\sigma t)$, where $\sigma = \frac{1}{\tau}$ and $\tau$ is some characteristic time needed for fusion. This concept, indeed, is more optmistic than the usual exponential stability, since in fusion research the main concern is whether a plasma can be confined long enough to achieve a self-sustaining thermonuclear reaction. On the other hand, many equilibria regarded as unstable by the criterion of exponential stability, are in fact $\sigma$-stable and may be found practically feasible for fusion research. A discussion of the energy principles may be found in Greene and Johnson (1968), Grossman (1968), and Spies (1976). The application of the $\sigma$-stability concept to various plasma configurations was further expounded by Goedbloed and Sakanaka (1973, I, II).

So far the modified energy principle, although well adopted in the study of MHD stability, has never been put on a rigorous basis since the question of the existence and uniqueness of a solution to the linearized MHD equations was never asked in the previous proofs. Without a precise answer to the question one cannot apply the energy principle with complete certainty. The first

attempt at the existence and uniqueness question was made by Rodriguez (1974). Several difficulties however were found in his proof. Among these, the connectivity of the vacuum region is not considered; the function space in which the Galerkin scheme is introduced is not complete; the scheme does not converge in the norm given (which in fact is only a pseudo norm). Moreover, some surface terms appearing in the variational formulation are discarded by questionable arguments. In fact, these terms cause grave concern in obtaining estimates. We, on the other hand, will make certian assumptions about these surface terms which characterize particular equilibrium states. Indeed, we doubt that solutions would exist for the linearized equations in the absence of such an assumption. The purpose of this paper is to give a correct and rigorous proof of the existence and uniqueness of a solution to the linearized equations governing a toroidal plasma, under a set of assumptions imposed upon an equilibrium state. In addition, the growth rates of the $L^2$ norms of both plasma displacement and velocity are obtained. In a subsequent report, Part II we shall justify the modified energy principle rigorously and a solid foundation is then laid for its application to problems of plasma stability.

In the following , we briefly describe the contents in each section. In Section 2, we formulate our problem based upon Lundquist equations for magnetohydrodynamics, and deal with the linearization of these equations. Since we consider the displacement of the confined plasma from its equilibrium configuration, it is natural to express the equations in terms of Lagrangian coordinates. Here we use an approach which is essentially that of Berstein et al. (1958) so that the final results may be checked with theirs. In the vacuum region, the magnetic field is both curl and divergence free. We follow Berstein et al. and introduce a vector potential instead of a scalar potential. The latter was used by Lüst and Martensen (1960) to study energy principles.

They showed that the two approaches are essentially equivalent provided certain cuts are made in the vacuum region and fluxes through these cuts are taken into consideration. In Section 3 a mixed boundary and initial value problem (MIBVP) is formulated for a toroidal plasma confined in a perfectly conducting shell. To ensure the uniqueness of the vector potential some flux conditions must be prescribed (Blank, Friedrichs and Grad, 1957), and a set of conditions on the equilibrium state is also precisely imposed. In Section 4 the Hodge decomposition theorem is used to define function spaces which play a fundamental role in this work. We then establish the continuity, coerciveness and symmetry of a bilinear form in Section 5. In Section 6 the MIBVP is recast in the form of an evolutionary variational problem (EVP) by means of the coercive bilinear form (Lions and Magenes 1972). We then construct an auxiliary EVP (AEVP) with zero flux conditions, whose solution is established by defining Galerkin approximations and showing they converge to a solution of AEVP. The solution of EVP is constructed by adding to the solution of AEVP a unique solution to an elliptic system, with inhomogeneous flux conditions. The uniqueness proof then follows easily from the coerciveness of the bilinear form and that of the elliptic system. As a consequence of the existence proof, the growth rates for the plasma displacement and kinetic energy are also obtained. In Section 7, we make some remarks regarding the solution of our problem and the relationship between a solution to EVP and a classical solution to MIBVP is discussed.

## §2. Linearization of Lundquist Equations.

We consider the following configuration for a confined plasma: $\Omega_p$ is a doubly connected bounded three dimensional region surrounded by a bounded, triply connected region $\Omega_v$. Thus $\Omega_p$ and $\Omega_v$ are topologically equivalent to two coaxial tori (Figure 1). The plasma, a highly ionized gas, is contained in $\Omega_p$. The vacuum region is surrounded by a conducting shell $\Gamma_v$.

In the plasma region $\Omega_p$, (Bernstein et al, 1957)

$$\rho \frac{dv}{dt} = - \nabla p + \mu J \times B, \tag{2.1}$$

$$\frac{\partial B}{\partial t} = - \nabla \times E, \tag{2.2}$$

$$\nabla \times B = \mu J, \tag{2.3}$$

$$\nabla \cdot B = 0, \tag{2.4}$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0, \tag{2.5}$$

$$\frac{d}{dt} (p\rho^{-\gamma}) = 0, \tag{2.6}$$

$$E + v \times B = 0. \tag{2.7}$$

Here,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla, \tag{2.8}$$

and all quantities are expressed in terms in Eulerian coordinates. $v$ is the fluid velocity, $B$ the magnetic field, $J$ the current density, $p$ the scalar fluid pressure, $\rho$ the mass density and $\gamma$ the adiabiatic constant, which may be taken to be equal to 5/3.

In the vacuum region $\Omega_v$,

$$\nabla \times B = 0, \tag{2.9}$$

$$\nabla \cdot B = 0, \tag{2.10}$$

the vacuum region is assumed free of currents or charges, and the fluid boundary is assumed to nowhere intersect the outer conducting shell.

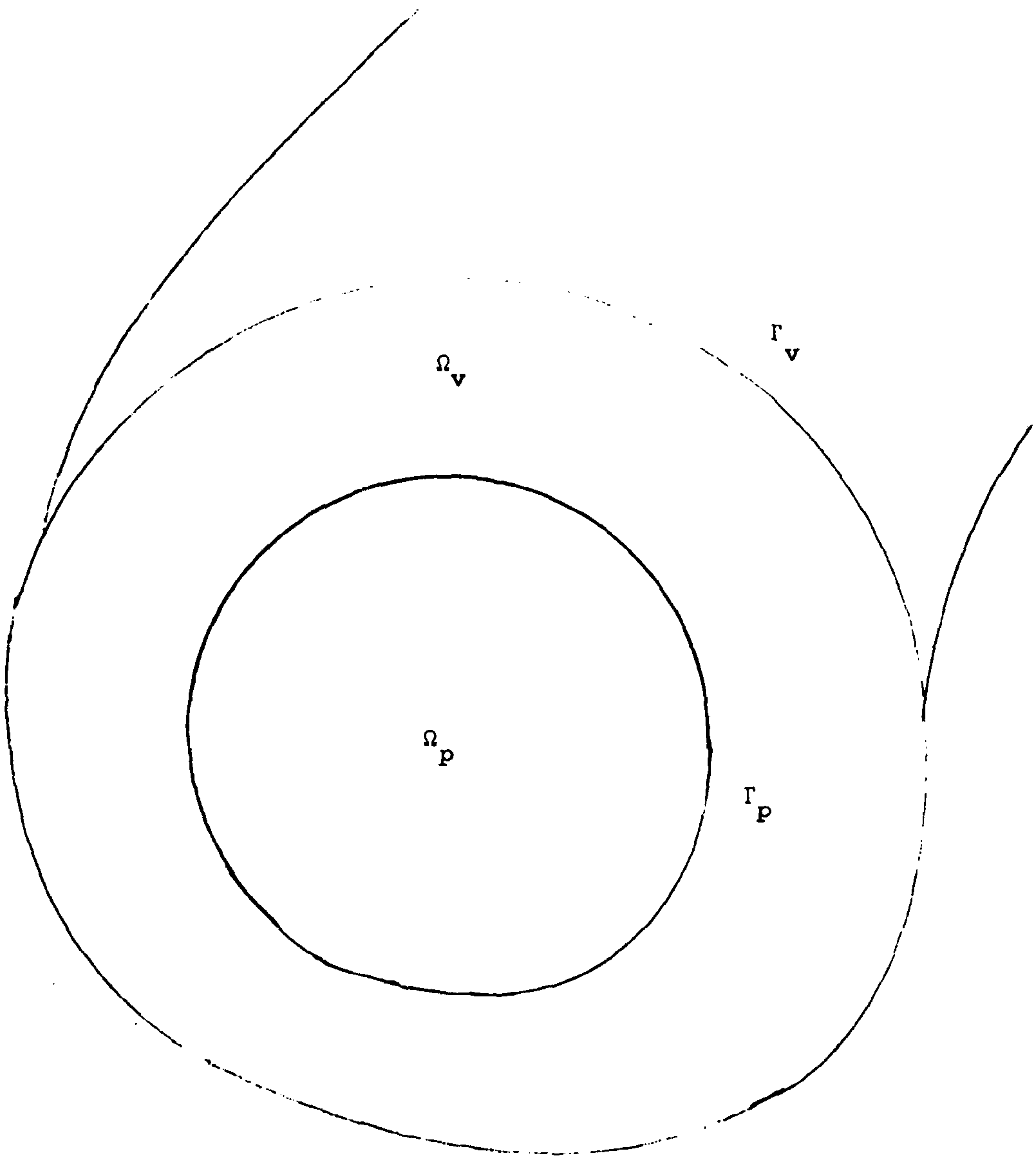


Figure 1

A Cross Section of the Toroidal Plasma

Furthermore, the two regions are coupled by a set of boundary conditions on $\Gamma_p$,

$$[p + \frac{B^2}{2\mu}] = 0, \tag{2.11}$$

$$[E + v \times B]_t = 0, \tag{2.12}$$

$$[B]_n = 0, \tag{2.13}$$

where [ ] designates the saltus of a quantity across the fluid-vacuum boundary, and the subscript t designates the tangential component.

On the conducting shell $\Gamma_v$:

$$E_t = 0,$$

$$B_n = 0.$$

In addition, since $\Omega_v$ is a multiply connected region, if the Pre-Maxwell equations in the vacuum region were solved independently, it would be necessary to prescribe certain "periods" of B and/or E in this region in order to ensure uniqueness of solutions. Physically, the system is linked to the interior through the metal shell. However, because of the coupling between $\Omega_p$ and $\Omega_v$, the situation is complicated, and we will not return to the question of prescribing periods until the linearization is completed. We also note that a detailed discussion of the formulation of the boundary conditions may be found in Blank, Friedrichs and Grad (1957). Also, appropriate units for various quantities are chosen so that $\mu = 1$.

We emphasize that the non-linear problem defined above is a "free boundary problem". $\Omega_p(t)$ and hence $\Gamma_p(t)$ a priori unknown must be determined as part of the solution. Therefore, we wish to examine the effect of small perturbations about an equilibrium state characterized by $v = 0$. This equilibrium state is governed by the equations:

In $\Omega_p$, $$\nabla p_0 = \nabla \times B_0 \times B_0, \tag{2.14}$$

$$\nabla \cdot B_0 = 0. \tag{2.15}$$

In $\Omega_v$, $$\nabla \times B_0 = J_0, \tag{2.16}$$

$$\nabla \cdot B_0 = 0, \tag{2.17}$$

with the boundary conditions:

$$[p_0 + \frac{B_0^2}{2}] = 0 \qquad \text{in } \Gamma_{p_0}, \tag{2.18}$$

$$B_0^v \cdot n = B_0^p \cdot n \qquad \text{on } \Gamma_{p_0}, \tag{2.19}$$

$$B_0^v \cdot n = 0 \qquad \text{on } \Gamma_{v_0}. \tag{2.20}$$

At equilibrium, we may assume $p = A(S)\rho^\gamma$ where $A(S)$ is a function of the entropy $S$. Above and hereafter a superscript $p$ or $v$ will designate plasma or vacuum variables respectively. In order to linearize our problem, we reformulate the problem in reference to the Lagrangian coordi-nates:

$$(x,y,z,t) \in \Omega_p \times R \xrightarrow[G]{} (x_0,y_0,z_0,t) \in \Omega_0 \times R,$$

where $$(G(x,y,z,t) = (x_0,y_0,z_0,t) + \varepsilon\xi(x_0,y_0,z_0t).$$

That is, we identify each fluid particle by its position in the equilibrium state and let $\varepsilon\xi(x_0,y_0,z_0,t)$ be the displacement of the fluid particle $x_0,y_0,z_0$ from its equilibrium state $x_0,y_0,z_0$ after a time $t$. We first express the operator $\nabla$ in Lagrangian coordinates. If $f(r,t)$ is given, where $r,t$ are the Eulerian variables and

$$f(r,t) = \tilde{f}(r_0,t),$$

where $r = (x,y,z)$ and $r_0 = (x_0,y_0,z_0)$, then it follows from the chain rule that

$$\nabla f(r,t) = \nabla_0\tilde{f} \cdot \nabla r_0.$$

Here $\nabla f$ and $\nabla_0 \tilde{f}$ are $1 \times 3$ column vectors, and

$$\nabla r_0 = \left( \frac{\partial r_0}{\partial x}, \frac{\partial r_0}{\partial y}, \frac{\partial r_0}{\partial z} \right) = \begin{bmatrix} \frac{\partial x_0}{\partial x} & \frac{\partial x_0}{\partial y} & \frac{\partial x_0}{\partial z} \\ \frac{\partial y_0}{\partial x} & \frac{\partial y_0}{\partial y} & \frac{\partial y_0}{\partial z} \\ \frac{\partial z_0}{\partial x} & \frac{\partial z_0}{\partial y} & \frac{\partial z_0}{\partial z} \end{bmatrix} .$$

Let

$$r_0 = r - \varepsilon\xi(r_0,t),$$

and express $\tilde{\xi}(r,t)$ by

$$\tilde{\xi}(r,t) = \xi(r_0,t),$$

or

$$\xi(r_0(r,t),t) = \tilde{\xi}(r,t).$$

Then

$$\nabla r_0 = I - \varepsilon\nabla_0\xi \cdot \nabla r_0,$$

where $\nabla_0\xi \cdot \nabla r_0$ denote the usual matrix multiplication, and it follows that

$$(I + \varepsilon\nabla_0\xi)\nabla r_0 = I. \qquad (2.21)$$

Formally, we may assume $(I + \varepsilon\nabla_0\xi)^{-1}$ exists to obtain

$$(I + \varepsilon\nabla_0\xi)^{-1} = \nabla r_0, \qquad (2.22)$$

and express

$$(I + \varepsilon\nabla_0\xi)^{-1} = I - \varepsilon\nabla_0\xi + \varepsilon^2(\nabla_0\xi)^2 + \cdots (-1)^{n+1}\varepsilon^n(\nabla_0\xi)^n + \cdots .$$

Without attempting to justify the convergence of this formula or its asymptotic validity, which would lead us to consider the validity of the linearization, we henceforth take

$$\nabla r_0 = (I + \varepsilon\nabla_0\xi)^{-1} = I - \varepsilon\nabla_0\xi, \qquad (2.23)$$

as an approximation.

We gather together the expressions for the other operations involving $\nabla$ as follows. To first order in $\varepsilon$,

$$\begin{aligned}\nabla \cdot v &= \text{Trace } (\nabla v) \\ &= \text{Trace } (\nabla_0 v \cdot (I + \varepsilon\nabla_0\xi)^{-1}) \\ &= \text{Trace } (\nabla_0 v(I - \varepsilon\nabla_0\xi)) \\ &= \nabla_0 \cdot v - \varepsilon \text{ Trace}(\nabla_0 v \cdot \nabla_0\xi)\end{aligned}$$

and

$$\begin{aligned}\nabla \times v &= [(I + \xi\nabla_0\xi)^{-1}\}^t \cdot \nabla_0] \times v \\ &= [(I - \varepsilon\nabla_0\xi)^t \cdot \nabla_0] \times v \\ &= [(\nabla_0 - \varepsilon(\nabla_0\xi)^t \cdot \nabla_0)] \times v,\end{aligned}$$

where $t$ denotes the transpose of the matrix. Furthermore, note that

$$\frac{\partial^e}{\partial t} + u \cdot \nabla^e = \frac{d^e}{dt} = \frac{\partial^\ell}{\partial t},$$

where superscripts $e$ and $\ell$ indicate Eulerian and Lagrangian coordinates respectively. Thus in Lagrangian coordiantes (2.1) to (2.7) become

$$\rho\frac{\partial v}{\partial t} = -\nabla_0 p \cdot [I + \varepsilon\nabla_0\xi]^{-1} + ([(I + \varepsilon\nabla_0\xi)^{-1^t}] \times B) \times B, \tag{2.24}$$

$$\frac{\partial B}{\partial t} = [(I + \varepsilon\nabla_0\xi)^{-1^t}\nabla_0] \times (u \times B_0) + (u \cdot \nabla_0)B \cdot (I + \varepsilon\nabla_0\xi)^{-1}, \tag{2.25}$$

$$\frac{\partial p}{\partial t} = -\gamma p \text{ Trace } \nabla_0 u \cdot (I + \varepsilon\nabla_0\xi)^{-1}. \tag{2.26}$$

Here the second and third equations were obtained by adding $u \cdot \nabla B$ and $u \cdot \nabla p$ respectively to both sides of the equation (2.3), and (2.26) is the equation which results from combining (2.6) and (2.7)

$$\frac{\partial p}{\partial t} = -u \cdot \nabla p - \gamma p\nabla \cdot u. \tag{2.27}$$

Also (2.5) becomes, in Lagrangian coordinates,

$$\frac{\partial \rho}{\partial t} = \rho \text{ Trace}[\nabla_0 \cdot u(I + \varepsilon\nabla_0\xi)^{-1}).$$

We now let

$$\rho = \rho_0 + \varepsilon\rho_1, \quad p = p_0 + \varepsilon p_1, \quad B = B_0 + \varepsilon B_1, \quad u = \varepsilon u_1.$$

Since zeroth order quantities are equilibrium quantities, we obtain the following system

$$\frac{\partial}{\partial t}\begin{bmatrix} \varepsilon u_1 \\ \\ \varepsilon B_1 \\ \\ \varepsilon p_1 \\ \varepsilon\rho_1 \\ \varepsilon\xi \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho_0+\varepsilon\rho_1}[-\nabla_0(p_0+\varepsilon p_1)(I-\varepsilon\nabla_0\xi\}] + \frac{1}{\rho_0+\varepsilon\rho_1}\{I-\varepsilon\nabla_0\})^t \cdot \\ \cdot\nabla_0\} \times (B_0+\varepsilon B_1) \times (B_0+\varepsilon B_1) \\ \{I-\varepsilon\nabla_0\xi\}^t \cdot \nabla_0 \times \{u_1 \times (B_0+\varepsilon B_1)\} + \\ u \cdot \nabla_0(B_0+\varepsilon B_1)(I-\varepsilon\nabla_0\xi) \\ -\gamma(p_0+\varepsilon p_1)\ \mathrm{Trace}\{\varepsilon\nabla_0 \cdot u_1(I-\varepsilon\nabla_0\xi)\} \\ (\rho_0+\varepsilon\rho_1)\ \mathrm{Trace}[\varepsilon\nabla_0\cdot u_1[I-\varepsilon\nabla_0\xi]] \\ \varepsilon u_1 \end{bmatrix}.$$

Keeping only first order terms in $\varepsilon$, that is equating coefficients of $\varepsilon$ on both sides, and using the equilibrium relation

$$\nabla_0 p_0 = \nabla_0 \times B_0 \times B_0 ,$$

we obtain

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ \\ p_1 \\ B_1 \\ \rho_1 \\ \xi \end{bmatrix} = \begin{bmatrix} -\frac{1}{p_0}(\nabla_0 p_1 - \nabla p_0 \nabla_0 u_1) + \frac{1}{p_0}(\nabla_0 \times B_1 \times B_0) \\ + \frac{1}{p_0}(\nabla_0 \times B_0 \times B_1) \\ -\gamma_i p_0 \nabla_0 \cdot u_1 \\ \nabla_0 \times (u_1 \times B_1) + u \cdot \nabla_0 B_0 \\ -\rho_0 \nabla_0 \cdot u_1 \\ u_1 \end{bmatrix} . \quad (2.28)$$

We next eliminate all quantities not directly expressed in terms of $u$ in order to obtain one equation for $u$, or $\xi$, which can be solved independently and then used to compute $p_1$, $B_1$, $\rho_1$. Alternately, we may differentiate the first equation in (2.1) with respect to time, assuming that spatial and time derivatives commute and in either case we obtain the same equation but in the latter case for $u$ rather than $\xi$. The equation for $\xi$ is

$$\rho_0 \frac{\partial^2}{\partial t^2}\xi = \nabla(\gamma p_0 \nabla_0 \cdot \xi) + \nabla p_0 \cdot \nabla_0 \xi + \nabla_0 \times \nabla_0 \times (\xi \times B_0) \times B_0$$
$$+ \nabla_0 \times (\xi \cdot \nabla_0 B_0) \times B_0 + \nabla_0 \times B_0 \times \nabla_0 \times (\xi \times B_0)$$
$$+ \nabla_0 \times B_0 \times (\xi \cdot \nabla_0 B_0) - (\nabla_0 \xi^T \cdot \nabla_0) \times B_0 \times B_0,$$

which reduces to the equation derived for $F(\xi)$ by Bernstein et al (1958)

$$\rho_0 \frac{\partial^2}{\partial t^2}\xi = \nabla_0(\gamma p_0 \nabla_0 \cdot \xi + \xi \cdot \nabla_0 p_0)$$
$$+ \nabla_0 \times \nabla_0 \times (\xi \times B_0) \times B_0 + \nabla_0 \times B_0 \times \nabla_0 \times (\xi \times B_0),$$

provided that

$$\nabla_0(\xi_0 \cdot \nabla_0 p_0) = \nabla_0 p_0 \cdot \nabla_0 \xi + \nabla_0 \times (\xi \cdot \nabla_0 B_0) \times B_0$$
$$+ \nabla_0 \times B_0 \times (\xi \cdot \nabla_0 B_0) - (\nabla_0 \xi^T \cdot \nabla_0) \times B_0 \times B_0,$$

the derivation of which is given in the appendix. To obtain the equation for $\xi$ by the first method, we integrate from 0 to $t$ the last four equations in our system to obtain

$$p_1(t) = -\gamma p_0 \nabla_0 \cdot \xi(t) + p_1(0) + \gamma p_0 \nabla_0 \cdot \xi(0),$$

$$B_1(t) = \nabla_0 \times (\xi(t) \times B_0) + \xi(t) \cdot \nabla_0 B_0$$
$$- \nabla_0 \times (\xi(0) \times B_0) - \xi(0) \cdot \nabla_0 B_0,$$

$$\rho_1(t) = -\rho_0 \nabla_0 \cdot \xi(t) + \rho_1(0) - \rho_0 \nabla_0 \cdot \xi(0).$$

Assuming that our system passes through the equilibrium state where $\xi = 0$, $p_1 = B_1 = 0$ at some time $t_0$, we may write

$$p_1(t) = -\gamma p_0 \nabla_0 \cdot \xi(t),$$

$$B_1(t) = \nabla_0 \times (\xi(t) \times B_0) + \xi(t) \cdot \nabla_0 B_0,$$

$$\rho_1(t) = -\rho_0 \nabla_0 \cdot \xi(t),$$

which, when we plug into the first equation of (2.28), together with the last equation in (2.28), again leads to (2.29).

The linearization in the vacuum region is done by introducing a vector potential $A$ there such that

$$E = E_0 + \varepsilon \dot{A}, \qquad B = B_0 + \varepsilon \nabla_0 \times A,$$

so that the Pre-Maxwell equations there are taken in the form

$$\nabla_0 \times \nabla_0 \cdot \dot{A} = 0, \qquad \nabla_0 \cdot \dot{A} = 0. \tag{2.29}$$

If we are dealing with the equation (2.29) written for $u$ instead of $\xi$, we retain (2.29). If not, we again assume the system passes through the equilibrium state at some time $t$ and integrate to obtain

$$\nabla_0 \times \nabla_0 \times A = 0, \qquad \nabla_0 \cdot A = 0.$$

For the boundary conditions:

$$(E + v \times B_0)_t = 0 \quad \text{or} \quad n \times (E + v \times B) = 0,$$

become to first order

$$n_0 \times (\dot{A} + \dot{\xi} \times B_0^v) = 0 \quad \text{i.e.} \quad n \times \dot{A} = (-n_0 \cdot \dot{\xi})B_0^v,$$

or in integrated form $n \times A = -(n \cdot \xi)B^v$, where we used $B_0^v \cdot n_0 = 0$, furthermore

$$p + \frac{B^{p^2}}{2} = \frac{B^{v^2}}{2} \quad \text{on} \quad \Gamma_p,$$

may be written as

$$(p_0 + \varepsilon p_1) + (\frac{B_0^p}{2} + \varepsilon\frac{B_1}{2})^2 = (\frac{B_0^v}{2} + \varepsilon(\nabla \times A))^2,$$

or,

$$(p_0 - \varepsilon\gamma p_0 \nabla \cdot \xi) + [\frac{(B_0^p)^2}{2} + \varepsilon(\nabla \times (\xi \times B_0) + \xi \cdot \nabla B_0)]$$

$$= (\frac{(B_0^v)^2}{2} + \varepsilon(\nabla \times A))^2,$$

which if we retain only terms of the first order in $\varepsilon$ reduces to:

$$-\gamma p_0 \nabla \cdot \xi + B_0^p \cdot (\xi \cdot \nabla B_0^p + \nabla \times (\xi \times B_0^p))$$

On $\Gamma_v$, we have $= B_0^v \cdot (\nabla \times A + \xi \cdot \nabla B_0^v)$.

$$E_t = 0, \quad \text{or} \quad n \times E = 0.$$

Hereafter we will omit the subscript 0 on equilibrium quantities.

It follows that

$$n \times \dot{A} = 0 \quad \text{or} \quad n \times A = 0 \quad \text{on} \quad \Gamma_v.$$

In summary, the equations derived apply either to pairs $(u_1, \dot{A})$ without restriction or to pairs $(\xi, A)$ when the system passes throught the equiblibrium state.

## §3. Variational Formulation of a Mixed Initial Boundary Value Problem (MIBVP).

We recapitulate the equations in §2 to define an MIBVP.

In $\Omega_p$,

$$\rho\ddot{\xi} = F(\xi) = \nabla(\gamma p \nabla \cdot \xi + \xi \cdot \nabla p) + \nabla \times B \times \nabla \times (\xi \times B) + \nabla \times \nabla \times (\xi \times B) \times B, \tag{3.1}$$

in $\Omega_v$,

$$\nabla \times \nabla \times A = 0, \tag{3.2}$$

$$\nabla \cdot A = 0, \tag{3.3}$$

subject to the following conditions:

(1) Boundary conditions.

On $\Gamma_p$,

$$L((\xi,A)) = 0 = -\gamma p \nabla \cdot \xi + B^p \cdot (\nabla \times (\xi \times B^p) + \xi \cdot \nabla B^p) - B^v \cdot (\nabla \times A + \xi \cdot \nabla B^v), \tag{3.4}$$

$$n \times A = -(n \cdot \xi) B^v. \tag{3.5}$$

On $\Gamma_v$,

$$n \times A = 0. \tag{3.6}$$

(2) Flux condition.

$$\int_{\Gamma_v} A(t) \cdot n \, ds = \sigma(t), \quad \sigma(t) \text{ prescribed.} \tag{3.7}$$

(3) Initial conditions.

$$\xi(0) = \xi_0, \quad \frac{\partial \xi}{\partial t}(0) = \dot{\xi}_0, \quad A(0) = A_0. \tag{3.8}$$

In a natural way, we shall reformulate the equations above in a variational form, which is more amenable to analysis. Its equivalence to the original formulation will be discussed later. We also note that in (3.8) the prescription of $A(0)$ is not necessary since it can be determined by $\xi(0)$.

We shall define a bilinear form on the product space $\Omega_p \times \Omega_v$, denoted by $a((\xi,A),(\tilde{\xi},\tilde{A}))$. To motivate the choice of this bilinear form and to make clear its connection with the original problem, we proceed as follows:

Given a pair $(\tilde{\xi},\tilde{A})$ defined on $\Omega_p \times \Omega_v$, we take the inner product of

$$\rho\ddot{\xi} = F(\xi),$$

with $\tilde{\xi}$, integrate over $\Omega_p$ and then integrate by parts to obtain

$$\langle \rho\ddot{\xi},\tilde{\xi}\rangle_{L^2(\Omega_p)} = \langle F(\xi),\tilde{\xi}\rangle_{L^2(\Omega_p)}$$

$$= \int_{\Omega_p} \{\nabla(\gamma p \nabla \cdot \xi + \xi \cdot \nabla p) + \nabla \times B \times (\nabla \times \xi \times B)$$

$$+ \nabla \times \nabla \times (\xi \times B) \times B\} \cdot \tilde{\xi}\, dV$$

$$= \int_{\Omega_p} -\{\gamma p \nabla \cdot \xi \nabla \cdot \tilde{\xi} + \nabla \times (\xi \times B) \cdot \nabla \times (\tilde{\xi} \times B)$$

$$- \tilde{\xi} \cdot \nabla \times B \times (\nabla \times (\xi \times B) - \tilde{\xi} \cdot \nabla(\xi \cdot \nabla p)$$

$$+ \int_{\Gamma_p} (\tilde{\xi} \cdot n)\{\gamma p \nabla \cdot \xi - B \cdot \nabla \times (\xi \times B\},$$

where use has been made of some well-known vector identities. In a similar fashion, we take the inner product in $\Omega_v$ of $\tilde{A}$ with the equation

$$\nabla \times \nabla \times A = 0,$$

and integrating by parts, we obtain

$$0 = +\int_{\Omega_v} \nabla \times A \cdot \nabla \times \tilde{A}\, dv + \int_{\Gamma_p} n \times \tilde{A} \cdot \nabla \times A\, dv + \int_{\Gamma_v} n \times \tilde{A} \cdot \nabla \times A\, dv.$$

If we assume that the pair $(\tilde{\xi},\tilde{A})$ satisfies the boundary conditions (3.5) and (3.6):

$$n \times \tilde{A} = -(n \cdot \tilde{\xi})B^v \quad \text{on } \Gamma_p,$$

$$n \times \tilde{A} = 0 \quad \text{on } \Gamma_v,$$

then we may write

$$0 = -\int_{\Omega_v} \nabla \times A \cdot \nabla \times \tilde{A}\, dv + \int_{\Gamma_p} (n \cdot \tilde{\xi})B^v \cdot \nabla \times A\, dv.$$

If we now simply add this result to the right-hand side of (3.7), we obtain

$$
\begin{aligned}
\langle\rho\ddot{\xi},\check{\xi}\rangle_{L^2(\Omega_p)} &= \int_{\Omega_p} -\{\gamma p(\nabla\cdot\xi)(\nabla\cdot\check{\xi}) \\
&- \check{\xi}\cdot\nabla(\xi\cdot\nabla p) + \nabla\times(\xi\times B)\cdot\nabla\times(\check{\xi}\times B) \\
&- \check{\xi}\cdot\nabla\times B\times(\nabla\times(\xi\times B))\}dv - \int_{\Omega_v}\nabla\times A\cdot\nabla\times\grave{A}\,dv \\
&+ \int_{\Gamma_p}[\check{\xi}\cdot n)[\gamma p\nabla\cdot\xi - B^p\cdot(\nabla\times(\xi\times B^p)) + \nabla\times A\cdot B^v]ds.
\end{aligned}
$$

Let

$$
\begin{aligned}
b((\xi,A),(\check{\xi},\grave{A})) &= \int_{\Omega_p}\{\gamma p(\nabla\cdot\xi)(\nabla\cdot\check{\xi}) - \check{\xi}\cdot\nabla(\xi\cdot\nabla p) \\
&+ \nabla\times(\xi\times B)\cdot(\nabla\times(\check{\xi}\times B)) - \check{\xi}\cdot\nabla\times B\times\nabla\times(\xi\times(\xi\times B)\}dv \\
&+ \int_{\Omega_v}\nabla\times A\cdot\nabla\times\grave{A}\,dv.
\end{aligned}
$$

and it follows that

$$
\begin{aligned}
&\langle\rho\ddot{\xi},\check{\xi})_{L^2(\Omega_p)} + b((\xi,A),(\check{\xi},\grave{A})) = \\
&\int_{\Gamma_p}(\check{\xi}\cdot n)(\gamma p\nabla\cdot\xi - B^p\cdot(\nabla\times(\xi\times B^p) + \nabla\times A\cdot B^v)ds.
\end{aligned}
$$

Note that the integrand in this boundary integral differs from the boundary condition of the MIBVP by

$$
\xi\cdot\frac{\nabla B^{v^2}}{2} - \xi\cdot\frac{\nabla B^{p^2}}{2}.
$$

So by defining

$$
a((\xi,A),(\check{\xi},\grave{A})) = b((\xi,A),(\check{\xi},\grave{A})) + \int_{\Gamma_p}(\check{\xi}\cdot n)\xi\cdot\nabla\left(\frac{B^{v^2}}{2} - \frac{B^{p^2}}{2}\right)ds, \quad (3.9)
$$

the above equation becomes

$$\langle \rho\ddot{\xi},\tilde{\xi}\rangle_{L^2(\Omega_p)} + a((\xi,A),\tilde{\xi},\tilde{A}))$$

$$= \int_{\Gamma_p} (\tilde{\xi} \cdot n)\{\gamma p \nabla \cdot \xi - B^p \cdot [\nabla \times (\xi \times B^p) + \nabla \times A \cdot B^v]$$

$$+ \xi \cdot \nabla(\frac{B^{v^2}}{2} - \frac{B^{p^2}}{2})\}ds$$

$$= \int_{\Gamma_p} (\tilde{\xi} \cdot n)L(\xi,A)ds. \qquad (3.10)$$

Thus we are led to define a solution of the evolutionary variational problem (EVP) associated with the MIBVP as a pair $(\xi,A)$ that satisfies

$$\frac{d^2}{dt^2} \langle \rho\xi,\tilde{\xi}\rangle_{L^2(\Omega_p)} + a((\xi,A),(\tilde{\xi},\tilde{A})) = 0 \ , \qquad (3.11)$$

for all $(\tilde{\xi},\tilde{A})$ in a space $W$ to be defined in the next section. After existence and uniqueness of a solution to (3.11) has been established, we shall return to the question of the equivalence of the EVP to the original problem.

Remark:

$a((\xi,A),(\tilde{\xi},\tilde{A}))$as defined by (3.9) will not be quite right for our purposes, however. It involves a term of the form $\int_{\Omega_p} \tilde{\xi} \cdot \nabla(\xi \cdot \nabla p)dv$ which will not necessarily make sense for $(\xi,A)$ in $W$, since $\xi$ may not be in $H'(\Omega_p)$. Thus the form of $a((\xi,A),(\tilde{\xi},\tilde{A}))$ used hereafter is obtained by integrating this troublesome term by parts to get, from (3.9),

$$a((\xi,A),(\tilde{\xi},\tilde{A})) = \int_{\Omega_p} \{\gamma p(\nabla \cdot \xi)(\nabla \cdot \tilde{\xi}) + (\nabla \cdot \tilde{\xi})(\xi \cdot \nabla p)$$

$$+ \nabla \times (\xi \times B) \cdot \nabla \times (\tilde{\xi} \times B) - \tilde{\xi} \cdot \nabla \times B \times \nabla \times (\xi \times B)\}dv$$

$$+ \int_{\Omega_v} \nabla \times A \cdot \nabla \times \tilde{A}\ dv + \int_{\Gamma_\pi} (\tilde{\xi} \cdot n)\xi \cdot \nabla(\frac{B^{v^2}}{2} - \frac{B^{p^2}}{2} - p)ds. \ (3.12)$$

For this form of $a((\xi,A),(\tilde{\xi},\tilde{A}))$, all terms as seen later will make sense when $(\xi,A)$ and $(\tilde{\xi},\tilde{A})$ belong to $W$, provided that the measure $d\mu = n \cdot \nabla(\frac{B_v^2}{2} - \frac{B_p^2}{2} - p)$ is either identically zero (no surface current) or nonnegative.

For later use, we have to make precise the smoothness conditions on the equilibrium state. Although a general existence theorem (or non-existence theorem) for a three dimensional equilibrium in the absence of symmetry is still lacking and there is good reason not to expect one except under special conditions, the assumptions below are not so restrictive in comparison with existing exact solutions. We assume that

$$p \in H^3(\Omega_p), \quad B \in H^3(\Omega_p), \quad B \in H^3(\Omega_v), \tag{3.13}$$

(for the definition of $H^n(\Omega)$ see §4) and $\Gamma_p$ and $\Gamma_v$ are of class $C^4$ and have the cone property. It then follows from the Sobolev embedding theorems (Lions and Magenes, 1972) that

$$p \in C^1(\Omega_p), \quad B \in C^1(\Omega_p), \quad B \in C^1(\Omega_v). \tag{3.14}$$

From the trace theorems we have (Lions and Magenes, 1972)

$$\nabla p \in H^{3/2}(\Gamma_p) \quad \text{and} \quad B \in H^{5/2}(\Gamma_p) \cap H^{5/2}(\Gamma_v),$$

which imply respectively

$$p \in C^1(\Gamma_p), \quad B \in C^2(\Gamma_p) \cap C^2(\Gamma_v). \tag{3.15}$$

We also assume that $n(x)$, the unit normal to $\Gamma_p$, can be extended in $\Omega_p$ as a function in $H^3(\Omega_p)$. This will be the case when $n(x) \in H^{5/2}(\Gamma_p)$ because of the inverse trace theorems. Consequently the embedding theorem implies that the extension of $n(x)$ has the properties

$$n(x), \quad \nabla(n(x)) \in C^0(\Omega_p). \tag{3.16}$$

Furthermore, the equilibrium is assumed to describe a truly sharp boundary in the sense that there exist constants $c_1, c_2, c_3, c_4, c_5, c_6$ such that:

$$0 < c_1 \leq p(x) \leq c_2 < \infty \quad \text{in} \quad \Omega_p , \tag{3.17}$$

$$0 < c_3 \leq \rho(x) \leq c_4 < \infty \quad \text{on} \quad \Omega_p , \tag{3.18}$$

$$0 < c_5 \leq |B^v| \leq c_6 < \infty \quad \text{on} \quad \Gamma_p . \tag{3.19}$$

Furthermore we assume that the equilibrium does *not* have any surface current, or that when there is a surface current present,

$$n \cdot \nabla(p + \frac{B^{p^2}}{2} - \frac{B^{v^2}}{2}) \leq 0 \quad \text{a.e. on} \quad \Gamma_p . \tag{3.20}$$

We conjecture that indeed this additional assumption is necessary in order that the dynamical equations be well posed. In what follows, we shall let

$$P = (p + \frac{(B^p)^2}{2} - \frac{(B^v)^2}{2}) .$$

## §4. Function Spaces

The object of this section is to introduce various function spaces, culminating in the introduction of a Pre-Hilbert space $W^-$ and an inner product $\langle\cdot,\cdot\rangle_{W^-}$ in this space. We shall show that the closure of the space $W^-$ in the norm $\|\ \|_{W^-}$, denoted by $W$, preserves certain important functional properties to justify its use in later sections as the fundamental space in which a solution to the EVP (3.11) is sought.

We shall denote by $H^1(\Omega_p)$ and $H^1(\Omega_v)$ respectively the spaces of vector valued functions $f = (F_1, f_2, f_3)$ from $\Omega_p$ and $\Omega_v$ into $R^3$ such that

$$\int_{\Omega_p} f_i^2\, dv + \int_{\Omega_p} (\nabla f_i)^2 dv < \infty \quad \forall i, \quad \text{for } H^1(\Omega_p),$$

$$\int_{\Omega_v} f_i^2\, dv + \int_{\Omega_v} (\nabla f_i)^2 dv < \infty \quad \forall i, \quad \textit{for } H^1(\Omega_v).$$

It is well known (Lions and Magenes, 1972) that the space $H^1(\Omega)$ is complete in the norm:

$$\|f\|_{H^1(\Omega)} = \{\sum_i \|f_i\|^2_{L^2(\Omega)} + \|\nabla f_i\|^2_{L^2(\Omega)}\}^{1/2}.$$

Here all derivatives are taken in the sense of distributions, and $\Omega$ denotes either $\Omega_p$ or $\Omega_v$. Similarly,

$$H^m(\Omega) = \{f \in L^2(\Omega) | D^\alpha f_i \in L^2(\Omega) \quad \text{for} \quad |\alpha| \leq m\} \qquad \forall i = 1,2,3,$$

with the norm:

$$\|f\|_{H^m(\Omega)} = (\sum_{i=1}^{i=3} \sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\Omega)})^{1/2}.$$

Next, to define a space $\bar{V}_2$ we first recall the Hodge decomposition theorem (Friedrichs, 1955). If

$$v \in L^2(\Omega),$$

then there exists $v_1 \in \bar{V}_1$, $v_2 \in \bar{V}_2$, $v_3 \in \bar{V}_3$ such that

$$v = v_1 + v_2 + v_3$$

or

$$L^2(\Omega) = \bar{V}_1 \oplus \bar{V}_2 \oplus \bar{V}_3 , \tag{4.1}$$

and the decomposition is orthogonal and thus unique, where

$$\bar{V}_1 = \{\nabla\phi \mid \phi \in H_0^1(\Omega)\}, \tag{4.2}$$

$\phi \in H_0^1$ means $\phi \in H^1$ and $\phi = 0$ on $\delta\Omega$ the boundary of $\Omega$,

$$\bar{V}_2 = \{\nabla \times v \mid v \in L^2(\mathrm{rot})(\Omega)\},$$

$$L^2(\mathrm{rot}) = \{v \in L^2(\Omega) \mid \nabla \times v \in L^2\ (\Omega)\}, \tag{4.3}$$

$$\bar{V}_3 = \{h \mid \nabla \times h = 0,\ \nabla \cdot h = 0 \text{ in } \Omega,$$

$$n \times h = 0 \text{ on } \delta\Omega\}. \tag{4.4}$$

Remark: If $v \in L^2(\Omega)$ has the property that

$$\nabla \cdot v = 0,$$

then

$$v = \nabla \times v_2 + h$$

where $\nabla \times v_2 \in \bar{V}_2$ and $h \in \bar{V}_3$. This is a consequence of

$$0 = \nabla \cdot v = \Delta\phi \Longrightarrow \phi = 0, \text{ since } \varepsilon \in H_0^1.$$

We will also need the space

$$L_\rho^2(\Omega_p) = \{\xi \mid \int_{\Omega_p} \rho\xi c^2 dx < \infty\}.$$

Note that given the assumption (3.16)

$$0 < c_3 \le \rho(x) \le c_4,$$

the space $L^2(\Omega_p)$ is both algebraically and topologically equivalent to $L^2(\Omega_p)$. For simplicity we shall denote the scalar product in $L_\rho^2(\Omega_p)$ by $\langle\cdot,\cdot\rangle_{2,\rho}$ , in $L^2(\Omega_p)$ by $\langle\cdot,\cdot\rangle_{2,p}$, and in $L^2(\Omega_v)$ by $\langle\cdot,\cdot\rangle_{2,v}$. The corresponding norms are denoted by $\|\ \|_{2,\rho}$, $\|\ \|_{2,p}$ and $\|\ \|_{2,v}$ respectively. If it is clear from the context, we shall use $L^2(\Omega_p)$ and $L^2(\Omega_v)$ to denote $L_2$-space of vector-valued functions or scalar functions. We shall also use $P_W(\Omega_p)$ to denote the projection of the set $W$ of pairs $(\xi,A)$ on $L^2(\Omega_p)$.

We now define the space $W^-$:

$$W^- = \begin{cases} (\xi,A) \mid \xi \in H^1(\Omega_p),\ A \in L^2(\mathrm{rot})(\Omega_v) \cap \bar{V}_2(\Omega_v) \\ \nabla \times \nabla \times A = 0,\ n \times A = (-n \cdot \xi)B^v \text{ on } \Gamma_p, \end{cases} \tag{4.5}$$

$$n \times A = 0 \quad \text{on} \quad \Gamma_v ,$$

and

$W$ = closure of $W^-$ in the inner product

$$\langle(\xi,A),(\tilde{\xi},\tilde{A})\rangle = \int_{\Omega_v} \nabla \times A \cdot \nabla \times \tilde{A}\, dv + \int_{\Omega_p} \gamma p \nabla \cdot \xi \nabla \cdot \tilde{\xi}\, dv$$

$$+ \int_{\Omega_p} \nabla \times (\xi \times B) \cdot \nabla \times (\tilde{\xi} \times B) db + \int_{\Gamma_p} (\xi \cdot n)(\tilde{\xi} \cdot n)(-n \cdot \nabla p) dx$$

$$+ \int_{\Omega_p} \rho \xi \cdot \tilde{\xi}\, dv. \tag{4.6}$$

To establish the functional properties of the elements in $W$, we need a *fundamental inequality which we shall first prove in the following*

Lemma: If $A \in L^2(\mathrm{rot}((\Omega) \cap \bar{V}_2(\Omega)$, where $\Omega$ may be multiply-connected with boundary $\Gamma$, then

$$\|A\|_2 \leq c\{\|\nabla \times A\|_2 + \|n \times A\|_{H^{-1/2}(\Gamma)}\}. \tag{4.7}$$

Proof:

By duality. We will show that when $A \in L^2(\mathrm{rot}) \cap \bar{V}_2(\Omega)$, for any $f \in L^2(\Omega)$, $A = (A_1,A_2,A_3)$, $f = (f_1,f_2,f_3$, then,

$$\left|\int A \cdot f\, dv\right| \leq (\|\nabla \times A\|_2 + \|n \times S\|_{H^{-1/2}(\Gamma)})\|f\|_2 .$$

For the proof, first note that we can reduce to the case where $f \in L^2 \cap \bar{V}_2(\Omega)$ since $A$ is orthogonal to the projections of $f$ into $\bar{V}_1$ and $\bar{V}_3$.

Let $f$ be so given, consider the auxiliary problem,

$$\begin{aligned} \nabla \times \nabla \times g &= f \\ \nabla \cdot g &= 0 \end{aligned} \quad \text{in } \Omega \qquad \Gamma = \bigcup_i \Gamma_i$$

$$n \times g = 0 \quad \text{on } \Gamma, \quad \int_{\Gamma_i} g \cdot n\, ds = 0.$$

Systems of this kind have been considered in the literature without proofs on specifications on $f$ by Blank, Friedrichs, Grad (1957), by Kress (1970) for the case of infinitely differentiable $f$, by Solonikoff (1978) in simply connected domains under general assumptions on $f$. For completeness and simplicity we give our own proof of the existence for the special cases considered. Let

$$H(\Omega)\left\{\begin{array}{l}\check{g} \subseteq L^2(\mathrm{rot})\Omega \cap \bar{V}_2(\Omega), \quad n \times \check{g} = 0 \quad \text{on} \quad \Gamma, \\ \langle g \cdot n, 1\rangle_{H^{1/2} \times H^{-1/2}(\Gamma)} = 0\end{array}\right\}.$$

$H$ is a Hilbert space for the inner product

$$\langle \check{g}, \check{}\rangle_H = \langle \nabla \times \check{g}, \nabla \times \check{h}\rangle_2 .$$

This follows from the Friedrichs inequality of the first kind in Friedrichs, (1955), or Werner (1968).

We now may consider the variational *problem* to find $g$ such that:

$$\int_\Omega \nabla \times g,\ \nabla \times \check{g}\ dv = \int_\Omega f \cdot \check{g}\ dv, \qquad \forall\ \check{g} \in H.$$

The Riesz representation theorem ensures us of a solution $g \in H$.

Furthermore

$$\nabla \times \nabla \times g = f.$$

To prove this, we need to show that

$$\int_\Omega g \cdot \nabla \times \nabla \times f\ dv = \int_\Omega f \cdot h\ dv, \qquad \forall\ h \in C_0^\infty(\Omega).$$

The validity of (4.9) follows from (4.8) for all $h \in C_0^\infty \cap H(\Omega)$ by integration by parts. To prove it for all $h \in C_0^\infty(\Omega)$, let $h = h^1 + h^2 + h^3$ be the decomposition of Hodge in which one always has $n \times h^1 = n \times h^3 = 0$. Since $h \in C_0^\infty(\Omega)$, $n \times h = 0$, therefore $n \times h^2 = 0$, and $h^2 \in H(\Omega)$. Thus since $h^1$ and $h^3$ contribute nothing to either side of (4.8) because of $\nabla \times h^1 = \nabla \times h^3 = 0$, $f \perp h^3$, holds for $\check{g} = h$ and also we obtain (4.9) again after integrating by parts.

Remark.

$$n \cdot \nabla \times g = 0 \quad \text{on} \quad \Gamma.$$

Note that $n \cdot \nabla\times$ is a first order tangential differential operator on $n \times g$, and therefore since $n \times g = 0$, the result follows.

Next we show that the estimate

$$\|g\|_{H^2(\Omega)} \leq c\|f\|_{2,\Omega},$$

holds. Here we appeal to the Friedrichs inequalities of the first and second kind (Friedrichs, 1955)

a) Since $g \in \bar{V}_2$ and $n \times g = 0$, we have

$$\|g\|_{\bar{H}'(\Omega)} \leq c\|\nabla \times g\|_{2,\Omega}.$$

b) Since $n \cdot \nabla \times g = 0$, $n \times g = 0$, $\nabla \cdot \nabla \times g = 0$,

$$\|\nabla \times g\|_{H'(\Omega)} \leq c\|\nabla \times \nabla \times g\|_{2,\Omega}.$$

The result now follows since

$$\nabla \times \left(\frac{\partial}{\partial x^i} g\right) = \frac{\partial}{\partial x^i}(\nabla \times g) \quad \text{in} \quad D'.$$

We now return to the proof of the Lemma,

$$\left|\int_\Omega A \cdot f\, dv\right| \leq \left|\int_\Omega A \cdot \nabla \times \nabla \times g\, dv\right|$$

$$\leq \left|\int_\Omega \nabla \times A \cdot \nabla \times g\, dv\right| + \left|\int_\Gamma n \times A \cdot \nabla \times g\, dv\right|$$

$$\leq \|\nabla \times A\|_{2,\Omega}\|\nabla \times g\|_{2,\Omega} + \|n \times A\|_{H^{-1/2}(\Gamma)}\|\nabla \times g\|_{H^{1/2}(\Gamma)}$$

$$\leq \|\nabla \times A\|_{2,\Omega}\|\nabla \times g\|_{2,\Omega} + c\|n \times A\|_{H^{-1/2}(\Gamma)}\|g\|_{H^2(\Omega)},$$

where for the last inequality we have used the trace theorem, $\|u\|_{H^{1/2}(\Gamma)} \leq c\|u\|_{H'(\Omega)}$, (Lion and Magenes, 1972) *given* the smoothness of $\Gamma$.

Thus since,

$$\|g\|_{H^2(\Omega)} \leq \|f\|_{2,\Omega},$$

and since obviously

$$\|\nabla \times g\|_{2,\Omega} \leq c\|g\|_{H^2(\Omega)},$$

there follows

$$|\int_\Omega A \cdot f\, dv| \leq c\{\|\nabla \times A\|_{2,\Omega} + \|n \times A\|_{H^{-1/2}(\Gamma)}\}\|f\|_{2,\Omega} .$$

This proves the Lemma.

Remark.

Note that if $(-n \cdot \nabla p) \geq c > 0$ a.e. on $\Gamma_p$, the proof would have been considerably simpler, since then the more elementary estimate,

$$\|A\|^2_{2,\Omega} \leq c\{\|\nabla \times A\|^2_{2,\Omega} + \|n \times A\|_{L^2(\Gamma)}\}$$

may be used.

<u>Theorem 4.1.</u>

$W$ is a Hilbert space of pairs $(\xi, A)$ satisfying

I. $\xi \in L^2(\Omega_p)$, $\nabla \times (\xi \times B) \in L^2(\Omega_p)$, $\nabla \cdot \xi \in L^2(\Omega_p)$.

II. $A \in L^2(\Omega_v)$, $\nabla \times A \in L^2(\Omega_v)$, $\nabla \times \nabla \times A = 0$.

III. $n \times A = (-n \cdot \xi)B^v$ on $\Gamma_p$, $n \times A = 0$, on $\Gamma_v$.

IV. $W$ is separable.

Proof:

I. We show $\xi \in L^2(\Omega_p) = L^2_\rho(\Omega_p)$.

For, assume that $(\xi^m, A^m) \in W^-$ is a Cauchy sequence in the $W^-$ norm, then by (4.6) $\|\xi_m\|_{2,\rho}$ is Cauchy.

$$\exists\, \xi \in L^2(\Omega_p), \quad \|\xi^m - \xi\|_{2,p} \to 0,$$

since $L^2(\Omega_p)$ is complete. It is also easy to see that $\xi \in L^2(\Omega_p)$.

Next we show that $\nabla \times (\xi \times B) \in L^2(\Omega_p)$.

If $(\xi^m, A^m) \in W^-$ is a Cauchy sequence, then by (4.6)

$$\|\nabla \times (\xi^m \times B)\|_{2,p}$$

is Cauchy, and

$$\exists G \in L^2(\Omega_p) \text{ such that } \|\nabla \times (\xi^m \times B) - G\|_{2,p} \to 0.$$

Also $\exists\xi^m \in L^2_\rho(\Omega_p)$ such that $\xi^m \times B \to \xi \times B$ in $\| \ \|_{2,\rho}$, and in particular $D'$. Hence

$$\nabla \times (\xi^m \times B) \xrightarrow{D'} \nabla \times (\xi \times B).$$

So by the uniqueness theorem for $D'$,

$$\nabla \times (\xi \times B) = G \in L^2(\Omega_p).$$

Finally we show $\nabla \cdot \xi \in L^2(\Omega_p)$.

We make use of the assumption (3.15) on the pressure. If $(\xi^m, A^m)$ is a Cauchy sequence in $\| \ \|_W$, then $\gamma_p \nabla \cdot \xi^m$ is Cauchy in $L^2(\Omega_p)$. Hence $\nabla \cdot \xi^m$ is Cauchy since $p$ is bounded below. Thus, by the same reasoning as above, $\nabla \cdot \xi$, taken in the sense of distribution, is in $L^2(\Omega_p)$.

II. $A \in L^2(\Omega_v)$ and $\nabla \times A \in L^2(\Omega_p)$, $\nabla \times \nabla \times A = 0$.

Assume $(\xi^m, A^m)$ converges in $\| \ \|_W$, then in particular there exists $W \in L^2(\Omega_v)$ such that $\nabla \times A^m \to W$ in $L^2(\Omega_v)$ and there exists $\xi \in L^2(\Omega_p)$ such that $\xi^m \to \xi$ in $L^2(\Omega_p)$, $\nabla \cdot \xi^m \to \nabla \cdot \xi$ in $L^2(\Omega_p)$. These last two results, together with the inequality (Temam, 1974):

$$\|\eta \cdot n\|_{H^{-1/2}(\Gamma)} \leq c\{\|\eta\|_{L^2(\Omega_p)} + \|\nabla \cdot \eta\|_{L^2(\Omega_p)}\} \tag{4.10}$$

imply

$$\xi^m \cdot n \to \xi \cdot n \text{ in } H^{-1/2}(\Gamma_p).$$

Therefore since by assumption $B^v$ is in $C^1(\Gamma_p)$,

$$(\xi \cdot n)B^v \in H^{-1/2}(\Gamma_p),$$

for $\phi \in L^2(\Gamma_p)$, dense in $H^{-1/2}(\Gamma_p)$, we have

$$\langle \xi \cdot n\, B^v, \phi \rangle_{H^{1/2} \times H^{-1/2}(\Gamma_p)} = \int_\Gamma (\xi \cdot m\, B^v) \cdot \phi \, ds$$
$$= \int_\Gamma (\xi \cdot m)\, B^v \cdot \phi \, ds.$$

Note that

$$\int_\Gamma (\xi \cdot m)\, B^v \cdot \phi \, ds \leq \|\xi \cdot m\|_{H^{-1/2}(\Gamma_p)} \|B^v \cdot \phi\|_{H^{1/2}(\Gamma)}$$
$$\leq c\|\xi \cdot m\|_{H^{-1/2}(\Gamma_p)} \|\phi\|_{H^{1/2}(\Gamma_p)}.$$

Hence also

$$(\xi^m \cdot m)B^v \to (\xi \cdot m)B^v \quad \text{in} \quad H^{-1/2}(\Gamma_p).$$

Now since,

$$n \times A^m = (\xi^m \cdot n)B^v,$$

$$n \times A^m \text{ is Cauchy in } H^{-1/2}(\Gamma_p).$$

Therefore the statement $A^m \to A$ in $L^2(\Omega_v)$ follows by the fundamental inequality (4.7)

$$\|A\|_{2,v} \le c\{\|\nabla \times A\|_{2,v} + \|n \times A\|_{H^{-1/2}(\Gamma)}\},$$

$$\forall A \in L^2(\mathrm{rot}(\Omega_v) \cap \bar{V}_2(\Omega_v).$$

Note that

$$\|n \times A\|_{H^{-1/2}(\Gamma)} = \|n \times A\|_{H^{-1/2}(\Gamma_p)},$$

since $\|n \times A\|_{H^{-1/2}(\Gamma_v)} = 0$ because of boundary condition (2.6).

III. $n \times A = (-n \cdot \xi)B^v$ on $\Gamma_p$, $n \times A = 0$ on $\Gamma_v$.

a) $(-n \cdot \nabla p) \ge c > 0$ a.e. on $\Gamma_p$, $(n \cdot \xi)^m$ is a Cauchy sequence in $L^2(\Gamma_p)$ directly from our definition of the norm $\|(\xi,A)\|_{\bar{W}}$, thus $(n \times A)^m$ is a Cauchy sequence in $L^2(\Gamma_p)$ since $B^v$ is smooth. Furthermore $(n \times A)^m$ is Cauchy in $L^2(\Gamma_v)$ since $n \times A^m = 0$ on $\Gamma_v$. Passing to the limit in the boundary condition, we have

$$n \times A = (-n \cdot \xi)B^v \quad \text{in} \quad L^2(\Gamma_p)$$

$$n \times A = 0 \quad \text{in} \quad L^2(\Gamma_v).$$

b) When $(-n \cdot \nabla p) = 0$ on $\Gamma_p$ on a set of non zero measure, we make use of the estimates (4.7) and (4.10) to conclude

$$(-n \cdot \xi)B^v = n \times A \quad \text{in} \quad H^{-1/2}(\Gamma_p)$$

$$n \times A = 0 \quad \text{in} \quad H^{-1/2}(\Gamma_v).$$

VI. Finally, we show that $W$ is separable.

This is so because $W$ can be embedded in a closed subspace of a product of $L^2$ spaces in the following way:

$$W \hookrightarrow L^2(\Omega_p) \times L^2(\Omega_p) \times L^2(\Omega_p) \times L^2(\Omega_v)$$

$$(\xi,A) \hookrightarrow (\xi, \gamma p \nabla \cdot \xi, \nabla \times (\xi \times B), \nabla \times A). \tag{4.11}$$

Here, we take as a natural norm on the product space, the one, whose square is the sum of the squares of the $L^2$ norms on each $L^2$ space.

Thus by virtue of the definition of $\| \ \|_W$ (4.6), this mapping (4.11) is clearly an isometry of $W$ onto its range, hence the range is closed. Thus $W$ is separable since any closed subspace of a Hilbert space is separable.

Another important fact necessary to understand the structure of the space $W$ is the following.

The vacuum member $A$ of a given element $(\xi,A) \in W$ is uniquely determined by the boundary values $\xi \cdot n$ on $\Gamma_p$. We formulate this statement as

<u>Theorem 4.2.</u>

There exists a unique solution of the system:

$$\nabla \times \nabla \times A = 0, \qquad \text{in} \quad \Omega_v ,$$

$$\nabla \cdot A = 0, \qquad \text{in} \quad \Omega_v ,$$

subject to

$$n \times A = -(n \cdot \xi)B^v \qquad \text{on} \quad \Gamma_p, \tag{4.12}$$

$$n \times A = 0 \qquad \text{on} \quad \Gamma_v ,$$

and

$$\int_{\Gamma_v} A \cdot n \, ds = 0.$$

Also, this solution $A$ is in $\bar{V}_2$. In particular, note that the only solution of

$$\nabla \times \nabla \times A = 0, \quad \nabla \cdot A = 0, \tag{4.13}$$

$$n \times A = 0 \text{ on } \Gamma_p \cup \Gamma_v, \quad \int_{\Gamma_v} A \cdot n \, ds = 0,$$

is the solution $A = 0$. The proof of this proposition is given in Kress (1970).

To conclude this section, we define three more spaces for later use:

$H$ : closure of $W$ (or $W^-$) in the norm:

$$\|(\xi,A)\|_H = \{\|\xi\|^2_{2,\rho} + \|A\|^2_{2,v}\}^{1/2}. \tag{4.14}$$

The spaces $W^+$ and $H^+$. Let

$$W^{-+} = \{(\xi,\hat{A}) | \xi \in P_{\Omega_p}(W^-), \hat{A} = A + A'\}$$

where

$$A \in P_{\Omega_v}(W^-), \quad A' \in \bar{V}_3 \tag{4.15}$$

then

$$W^+ = \text{closure of } W^{-+} \text{ in the } W \text{ norm.} \tag{4.16}$$

$$H^+ = \text{closure of } W^{-+} \text{ in the } H \text{ norm.} \tag{4.17}$$

## §5. Continuity, coerciveness and symmetry of $a((\xi,A), (\tilde{\xi},\tilde{A}))$.

The expression for $a((\xi,A),(\tilde{\xi},\tilde{A}))$ defined in §3 is given by

$$a((\xi,A),(\tilde{\xi},\tilde{A})) = \int_{\Omega_v} \nabla \times A \cdot \nabla \times \tilde{A}\, dv$$

$$+ \int_{\Omega_p} \{\nabla \times (\xi \times B) \cdot \nabla \times (\tilde{\xi} \times B) + (\nabla \cdot \tilde{\xi})(\xi \cdot \nabla p)$$

$$- \tilde{\xi} \cdot \nabla \times B \times (\nabla \times (\xi \times B)) + \gamma p(\nabla \cdot \xi)(\nabla \cdot \tilde{\xi}) dv$$

$$- \int_{\Gamma_p} (\tilde{\xi} \cdot n)(\xi \cdot n) n \cdot \nabla P ds\ , \qquad (5.1)$$

where

$$P = p + \frac{B^{p^2}}{2} - \frac{B^{v^2}}{2}\ .$$

We first prove that $a((\xi,A),(\tilde{\xi},\tilde{A}))$ is a continuous bilinear form in $W \times W$.

<u>Theorem 5.1.</u>

For any $(\xi,A),(\tilde{\xi},\tilde{A}) \in W$, there exists $c > 0$, a constant, such that

$$|a((\xi,A),(\tilde{\xi},\tilde{A}))| \leq c\|(\xi,A)\|_W\|(\tilde{\xi},\tilde{A})\|_W\ . \qquad (5.2)$$

Remark:

Hereafter we shall use $c$ as a generic positive constant.

Proof: The integrals on the right hand side of (5.1) are estimated as follows:

$$\int_{\Omega_v} \nabla \times A \cdot \nabla \times \tilde{A}\, dv \leq \{\int_{\Omega_v} |\nabla \times A|^2 dv\}\{\int_{\Omega_v} |\nabla \times \tilde{A}|^2 dv\}^{1/2}\ . \qquad (5.3)$$

$$\int_{\Omega_p} \gamma p \nabla \cdot \xi \nabla \cdot \tilde{\xi}\, dv \leq \{\int_{\Omega_p} \gamma p(\nabla \cdot \xi)^2 dv\}^{1/2} \{\int_{\Omega_p} \gamma p(\nabla \cdot \tilde{\xi})^2 dv\}^{1/2}\ . \qquad (5.4)$$

$$\int_{\Omega_p} \nabla \times (\xi \times B) \cdot \nabla \times (\tilde{\xi} \times B) dv$$

$$\leq \{\int_{\Omega_p} |\nabla \times (\xi \times B)|^2 dv\}^2 \{\int_{\Omega_p} |\nabla \times (\tilde{\xi} \times B)|^2 dv\}^{1/2}\ . \qquad (5.5)$$

$$\int_{\Omega_p} |\tilde{\xi} \cdot \nabla \times B \times (\nabla \times (\xi \times B)| dv$$

$$\leq \{\int_{\Omega_p} \rho\tilde{\xi}^2 dv\}^{1/2} \{\int_{\Omega_p} \frac{(\nabla \times B)^2}{\rho} |\nabla \times (\xi \times B)|^2 dv\}^{1/2} \tag{5.6}$$

$$\leq \{\int_{\Omega_p} \rho\tilde{\xi}^2 dv\}^{1/2} \{c \int_{\Omega_p} |\nabla \times (\xi \times B)|^2 dv\}^{1/2}. \tag{5.7}$$

$$\int_{\Omega_p} (\nabla \cdot \tilde{\xi})(\xi \cdot \nabla_p) dv$$

$$\leq (\int_{\Omega_p} \gamma p(\nabla \cdot \tilde{\xi})^2 dv)(\int_{\Omega_p} \frac{(\nabla p)^2}{\gamma p \rho} \rho\xi^2 dv) \tag{5.8}$$

$$\leq c(\int_{\Omega_p} \gamma p(\nabla \cdot \tilde{\xi})^2 dv)(\int_{\Omega_p} \rho\xi^2 dv). \tag{5.9}$$

$$\int_{\Gamma_p} (\tilde{\xi} \cdot m)(\xi \cdot n)(-n \cdot \nabla p) ds$$

$$\leq \int_{\Gamma_p} (\tilde{\xi} \cdot m)^2 (-n \cdot \nabla p) ds \int_{\Gamma_p} (\xi \cdot m)^2 (-n \cdot \nabla p) ds, \tag{5.10}$$

where

$$-n \cdot \nabla p \geq 0 \quad \text{a.e.} \quad \text{on} \quad \Gamma_p.$$

In (5.3) to (5.9) the essential tool is the Schwartz inequality. In (5.4) we apply it after writing $\gamma p = \sqrt{\gamma p}\,\sqrt{\gamma p}$. In (5.6) and (5.8) we first multiply and divide by $\sqrt{\rho}$. In (5.8) we multiply and divide by $\sqrt{\gamma p}$. In (5.10) we use the Schwartz inequality after writing $-n \cdot \nabla p = \sqrt{-n \cdot \nabla p} \cdot \sqrt{-n \cdot \nabla p}$.

Thus (5.3) through (5.10) combine to yield

$$|a((\xi,A),(\tilde{\xi},\tilde{A}))| \leq c\{\|\sqrt{\gamma p}\, \nabla \cdot \xi\|_{2,p} \|\sqrt{\gamma p}\, \nabla \cdot \tilde{\xi}\|_{2,p}$$

$$+ \|\nabla_x (\xi \times B)\|_{2,p} \|\nabla \times (\tilde{\xi} \times B)\|_{2,p} + \|\sqrt{\gamma p}\, \nabla \cdot \tilde{\xi}\|_{2,p} \|\xi\|_{2,p}$$

$$+ \|\nabla \times (\xi \times B)\|_{2,p}^2 \|\xi\|_{2,p} + \{\int_{\Gamma_p} (\xi \cdot n)^2 (-n \cdot \nabla p) ds\}\{\int_{\Gamma_p} (\tilde{\xi} \cdot m)^2 (-n \cdot \nabla p) ds\}.$$

Therefore using Schwarz's inequality,

$$\sum a_i b_i \le (\sum a_i^2)^{1/2} (\Omega\, b_i^2)^{1/2} ,$$

we obtain,

$$|a((\xi,A),(\tilde{\xi},\tilde{A}))| \le c\{\|\sqrt{\gamma p}\ \nabla \cdot \xi\|^2_{2,p} + \|\nabla \times (\xi \times B)\|^2_{2,p} + \|\varepsilon\|^2_{2,\rho}$$

$$+ \int_{\Gamma_p} (\xi \cdot m)^2 ds\}\{\|\sqrt{\gamma p}\ \nabla \cdot \tilde{\xi}\|^2_{2,p} + \|\nabla \times (\tilde{\xi} \times B)\|^2_{2,p}$$

$$+ \|\tilde{\xi}\|^2_{2,p} + \int_{\Gamma_p} (\tilde{\xi} \cdot m)^2 ds\} \le c\|(\xi,A)\|_W \|(\tilde{\xi},\tilde{A})\|_W .$$

This completes the proof of the theorem.

Next we show that under the assumptions (3.13) to (3.18), $a((\xi,A),(\tilde{\xi},\tilde{A}))$ is coercive.

<u>Theorem 5.2.</u>

There exist positive constants $\delta$ and $\lambda$ such that:

$$a((\xi,A),(\xi,A)) + \lambda\|\xi\|^2_{L^2_\rho(\Omega_p)} \ge \delta\|(\xi,A)\|^2_W , \tag{5.11}$$

where

$$a((\xi,A),(\xi,A)) = \int_{\Omega_p} \{\gamma p(\nabla \cdot \xi)^2 + [\nabla \times (\xi \times B)]^2$$

$$+ (\nabla \cdot \xi)(\xi \cdot \nabla p) - \xi \cdot \nabla \times B \times \nabla \times (\xi \times B)\}dv$$

$$\| \int_{\Omega_v} (\nabla \times A)^2 dv - \int (\xi \cdot n)^2 n \cdot \nabla p\ ds .$$

An integration by parts of the volume terms, first shown by Kruskal et al. (1958), justifies the identity (Spies, 1974):

$$\int_{\Omega_p} \{\gamma p(\nabla \cdot \xi)^2 + [\nabla \times (\xi \times B)]^2 + (\nabla \cdot \xi)(\xi \cdot \nabla p)$$

$$- \xi \cdot \nabla \times B \times \nabla \times (\xi \times B)\}dv$$

$$= \int_{\Omega_p} [\nabla \times (\xi \times B) + n \cdot \xi \nabla \times B \times n]^2 + \gamma p(\nabla \cdot \xi)^2$$

$$- 2(\nabla \times B) \times n \cdot (B \cdot \nabla n)(\xi \cdot n)^2 dv .$$

Here the only possibly negative quantity is

$$-\int_{\Omega_p} 2(\nabla \times B) \times n \cdot (B \cdot \nabla n)(\xi \cdot n)^2 dv, \tag{5.12}$$

and

$$n = \frac{\nabla p}{|\nabla p|} .$$

Letting

$$2(\nabla \times B) \times n \cdot (B \cdot \nabla n)(x) = M(x),$$

we may find a constant $e$ such that

$$-\int_{\Omega_p} M(\xi \cdot n)^2 dv + e\int_{\Omega_p} \rho\xi^2 dv \geq 0 ,$$

$$\forall\xi, \text{ such that } (\xi,A) \in W.$$

Since

$$\xi^2 = (n \cdot \xi)^2 + (n \times (n \times \xi))^2 ,$$

$$\rho(\xi \cdot n)^2 \leq \rho\xi^2 ,$$

it suffices to let

$$e = \left\|\frac{M}{\rho}\right\|_{L^\infty} = \left\|\frac{2\nabla \times B \times n \cdot B \cdot \nabla n}{\rho}\right\|_{L^\infty} . \tag{5.13}$$

So by integration by parts, we obtain

$$a((\xi,A),(\xi,A)) + e\|\xi\|^2_{2,\rho} \geq 0 , \tag{5.14}$$

$$\forall(\xi,A) \in W.$$

(5.14) will prove a useful result when we analyze the stability of solutions to the MIBVP; however, it is not sufficient to establish coerciveness as it is not possible in this approach to obtain $\delta\|(\xi,A)\|^2_W$ on the R.H.S. of (5.11). This can, however, be done in the following way:

$$
\begin{aligned}
&\left|\int_{\Omega_p} \xi \cdot \nabla \times B \times \nabla \times (\xi \times B) dv\right| \\
&\leq \int_{\Omega_p} |\xi||\nabla \times B||\nabla \times (\xi \times B)| dv \\
&\leq e \int_{\Omega_p} |\xi||\nabla \times (\xi \times B)| dv \\
&\leq e_1 \{\int_{\Omega_p} |\xi|^2\}^{1/2} \{\int_{\Omega_p} |\nabla \times (\xi \times B))^2\}^{1/2} \\
&\leq e_1 \{\frac{1}{4f_1} \int_{\Omega_p} |\xi|^2 dv + f_1 \int_{\Omega_p} |\nabla \times (\xi \times B)|^2 dv \\
&\leq e_1 \{\frac{e_2}{4f_1} \|\xi\|^2_{2,\rho} + f_1 \|\nabla \times (\xi \times B)\|^2_{2,p}\},
\end{aligned}
$$

where we have used the inequality

$$ab \leq \frac{1}{4f} a^2 + fb^2 ,$$

and

$$e_1 = \|\nabla \times B\|_{L^\infty} , \quad e_2 = [g1b\ \rho(x)]^{-1} .$$

In the same way, we obtain that, for arbitrary $f_2$ ,

$$\left|\int_{\Omega_p} (\nabla \cdot \xi)(\xi \cdot \nabla p) dv\right| \leq e_3 \{\frac{e_2 \|\xi\|^2_{2,\rho}}{4f_2} + f_2 e_4 \|\sqrt{\gamma p}\ \nabla \cdot \xi\|^2_{2,\rho}\} ,$$

where

$$e_3 = \|\nabla p\|_{L^\infty} , \quad e_4 = [g1b\gamma p]^{-1} .$$

The surface contribution in the expression for $a((\xi,A),(\xi,A))$ is

$$-\int_{\Gamma_p} (\xi \cdot n)^2 n \cdot \nabla\left(p + \frac{B^{p^2}}{2} - \frac{B^{v^2}}{2}\right) dv$$

which is nonnegative, by assumption.

We now piece together the above estimates and find

$$a((\xi,A),(\xi,A)) + \left\{\frac{e_1e_2}{4f_1} + \frac{e_3e_2}{4f_2} + 1\right\}\|\xi\|^2_{2,\rho}$$

$$\geq (1 - e_1f_1)\|\nabla \times (\xi \times B)\|^2_{2,p} + (1 - e_3e_4f_2)\|\sqrt{\gamma p}\ \nabla \cdot \xi\|^2_{2,\rho}$$

$$+ \int_{\Gamma_p} (\xi \cdot n)^2(-n \cdot \nabla p)ds + \|\nabla \times A\|^2_{2,v} + \|\xi\|^2_{2,\rho} .$$

By choosing $f_1$ and $f_2$ for instance so as to satisfy

$$e_1f_1 = 1/2 \qquad e_3e_4f_2 = 1/2\ ,(5.15)$$

we thus obtain from (5.15)

$$a((\xi,A),(\xi,A)) + \lambda\|\xi\|^2_{2,\rho} \geq \delta\|(\xi,A)\|^2_W$$

with

$$\delta = 1/2\ , \quad \lambda = \frac{e_1^2e_2}{2f_1} + \frac{e_3^2e_2e_4}{2f_2} + 1. \tag{5.16}$$

Finally we establish the symmetry of $a((\xi,A),(\tilde{\xi},\tilde{A}))$.

<u>Theorem 5.3.</u>

*For any pair* $(\xi,A) \in W^+$, $(\tilde{\xi},\tilde{A}) \in W^+$ ,

$$a((\xi,A), (\tilde{\xi},\tilde{A})) = a((\tilde{\xi},\tilde{A}),(\xi,A)).$$

Proof:

From (5.1), for $(\xi,A) \in W^{-+}$, $(\tilde{\xi},\tilde{A}) \in W^{-+}$

$$a((\xi,A),(\tilde{\xi},\tilde{A})) - a((\tilde{\xi},\tilde{A}),(\xi,A)) =$$

$$\int_{\Omega_p} \{\tilde{\xi} \cdot \nabla(\xi \cdot \nabla p) - \xi \cdot \nabla(\tilde{\xi} \cdot \nabla p) + \xi \cdot \nabla \times B \times \nabla \times (\tilde{\xi} \times B)$$
$$- \tilde{\xi} \cdot \nabla \times B \times \nabla \times (\xi \times B)\}dv. \tag{5.17}$$

If we can show that this expression is zero, then since $W^{-+}$ is dense in $W^+$, by passing to the limit in pairs $(\xi^m,A^m) \in W^{-+} \xrightarrow{W} (\xi,A) \in W^+$, $(\tilde{\xi}^m,\tilde{A}^m) \in W^{-+} \xrightarrow{W} (\tilde{\xi},\tilde{A}) \in W^+$, the proof of symmetry of $W^+$ will be complete.

We will show that the expression above can be expressed as a divergence which in turn can be converted into a surface term which vanishes.

Now we establish

$$\int_{\Omega_p} \{-\tilde{\xi} \cdot \nabla(\xi \cdot \nabla_p) + \xi \cdot \nabla(\tilde{\xi} \cdot \nabla p) + \xi \cdot \nabla \times B \times \nabla \times (\tilde{\xi} \times B)$$

$$- \tilde{\xi} \cdot \nabla \times B \times \nabla \times (\xi \times B)\}dv$$

$$= + \int_{\Omega_p} \nabla \cdot \{B \cdot (\tilde{\xi} \times \xi)J\}dv \ .$$

For this, we show the following identity holds:

$$B(\xi,\tilde{\xi}) = -\tilde{\xi} \cdot \nabla(\xi \cdot \nabla p) + \xi \cdot \nabla(\tilde{\xi} \cdot \nabla p)$$

$$+ \xi \cdot \nabla \times B \times \nabla \times (\tilde{\xi} \times B) - \tilde{\xi} \cdot \nabla \times B \times \nabla \times (\xi \times B)$$

$$- \nabla \cdot (B \cdot (\tilde{\xi} \times \xi)J) = 0 \tag{5.18}$$

for $\xi,\tilde{\xi} \in P_{\Omega_p}(W^{-+})$, the set of all first components of pairs $(\xi,A) \in W^{-+}$. Note that

$$B(\xi + \xi',\tilde{\xi} + \tilde{\xi}') = B(\xi,\tilde{\xi}) + B(\xi',\tilde{\xi}) + B(\xi,\tilde{\xi}') + B(\xi',\tilde{\xi}'). \tag{5.19}$$

If the pair $(\xi,\tilde{\xi})$, $(\xi',\tilde{\xi})$, $(\xi,\tilde{\xi}')$, $(\xi',\tilde{\xi}')$ satisfy the identity, then the pairs $(\xi + \xi',\tilde{\xi} + \tilde{\xi}')$ also do. Also note that if either the first or second member in the pair $(\xi,\tilde{\xi})$ is zero, the identity is trivially satisfied.

To establish (5.18), we need the Lemmas below.

<u>Lemma 5.1.</u>

$$B(\xi,\tilde{\xi}) = 0 \quad \text{implies} \quad B(\alpha(x)\xi,\tilde{\xi}) = 0 \ ,$$

$$\forall \quad \text{scalar} \quad \alpha(x). \tag{5.20}$$

Proof:

$$(\tilde{\xi} \cdot \nabla\alpha)(\xi \cdot \nabla p) - \nabla \times B \cdot \tilde{\xi} \times ((\nabla\alpha) \times (\xi \times B))$$

$$= (\nabla \times B \cdot \nabla\alpha)(B \cdot \xi \times \tilde{\xi}) \ .$$

That this last equation is indeed an identity is immediate from the relations:

$$(\tilde{\xi} \cdot \nabla\alpha)(\xi \cdot \nabla p)$$

$$= (\tilde{\xi} \cdot \nabla\alpha)(\xi \cdot \nabla \times B \times B) - \nabla \times B \cdot \tilde{\xi} \times (\nabla\alpha \times (\xi \times B))$$

$$= \nabla \times B \cdot (\tilde{\xi} \cdot (\tilde{\xi} \times B)\nabla\alpha - (\tilde{\xi} \cdot \nabla\alpha)(\xi \times B))$$

$$= (\nabla \times B \cdot \nabla\alpha)(\tilde{\xi} \cdot (\tilde{\xi} \times B) + (\tilde{\xi} \cdot \nabla\alpha) \cdot (\xi \cdot \nabla \times B \times B)$$

$$= -(\nabla \times B \cdot \nabla\alpha)(B \cdot (\xi \times \tilde{\xi}) + (\tilde{\xi} \cdot \nabla a)(\xi \cdot \nabla \times B \times B).$$

Lemma 5.2.

The identity (5.18) holds for the Cartesian basis vectors

$$\xi = e_i \ , \quad \tilde{\xi} = e_j \ , \quad i,j = 1,2,3.$$

The proof of Lemma 5.2 is straightforward but tedious and we omit it here. Lemmas 5.1 and 5.2 together with (5.19) combined with lineariy and density of $W^{-+}$ in $W^{+}$ establish Theorem 5.3 .

## §6. Existence of a Solution to EVP.

In this section, we establish the existence of a solution to the following EVP:

In the interval $[0,T]$, $T > 0$, find $(\xi,\hat{A})(t) \in W^+$ such that

$$\frac{d^2}{dt^2} \langle \xi(t), \tilde{\xi} \rangle_{2,\rho} + a((\xi,\hat{A})(t),(\tilde{\xi},\tilde{\hat{A}})) = 0 \ ,$$

$$\forall(\tilde{\xi},\tilde{\hat{A}}) \in W^+ \tag{6.1}$$

subject to the initial conditions

$$(\xi(0),\hat{A}(0)) = (\xi_0,\hat{A}_0) \in W^+ \ , \tag{6.2}$$

$$\frac{\partial \xi}{\partial t}(0) = \dot{\xi}_0 \in P_{\Omega_p}(H), \tag{6.3}$$

and the flux condition

$$\int_{\Gamma_v} \hat{A}(t) \cdot n \, ds = \sigma(t), \tag{6.4}$$

where $\sigma(t)$ is a given continuous function of $t$ on $[0,T]$. The problem will be divided into two stages. Much of the work consists in defining a solution to an auxiliary evolutionary variational problem (AEVP) and proving the existence and uniqueness of a solution to the AEVP. Then we shall add to the vacuum part of the solution of this auxiliary problem an appropriate solution of an elliptic system of equations in order to obtain the actual solution to the EVP.

### §6.1. An Auxiliary Evolutionary Variational Problem (AEVP).

The AEVP is formulated as follows:

On $[0,T]$, find $(\xi,A)(t) \in C([0,\infty):W)$ with $\dot{\xi}(t) \in C([0,\infty):P_{\Omega_p}(H))$

such that

$$\frac{d^2}{dt^2} \langle \xi(t),\tilde{\xi} \rangle_{2,\rho} + a((\xi,A)(t),(\tilde{\xi},\tilde{A})) = 0 \ ,$$

$$\forall(\tilde{\xi},\tilde{A}) \in W, \tag{6.5}$$

subject to the initial conditions

$$(\xi(0),A(0)) = (\xi_0,\hat{A}_0 - A_0') \in W, \tag{6.6}$$

$$\frac{\partial\xi}{\partial t}(0) = \dot{\xi}_0 \in P_{\Omega_p}(H^+) = P_{\Omega_p}(H). \tag{6.7}$$

where specification of $\dot{A}(0)$ is not required, and $A_0'$ is an element of $\bar{V}_3$ satisfying

$$\int_{\Gamma_v} A' \cdot n \, ds = \sigma(0), \tag{6.8}$$

and the flux condition

$$\int_{\Gamma_v} A \cdot n \, ds = 0. \tag{6.9}$$

Note that this flux condition actually follows (cf. §4) from $A \in P_{\Omega_v}(W)$, but we state it explicitly here for the sake of convenience.

To show the existence of a solution to the AEVP, we shall define a sequence of Galerkin approximations, and then use the coerciveness of $a((\xi,A),(\xi,A))$ to derive an energy inequality. This energy inequality will then be used to show that the solutions are strongly bounded in a suitable space, and hence converge weakly. Finally we shall show that the limit is indeed a solution of AEVP. In order to define the Galerkin approximations, we begin by proving the following

<u>Lemma 6.1:</u> There exists a sequence of elements $\beta^i = (\beta_1^i,\beta_2^i)$ that satisfy the following conditions:

(1) $\{\beta^i\}_{i=1}^{\infty}$ constitutes a basis in $H$ and

$\langle\beta_1^i,\beta_1^j\rangle_{2,\rho} = \delta_{ij}$ ,

(2) $\{\beta^i\}_{i=1}^{\infty}$ form a complete set in $W$, i.e. finite linear combinations of elements of the sequence are dense in $W$.

Proof: Since $W$ is separable, there exists a sequence of linearly independent vectors $\{\alpha^i\}_{i=1}^{\infty}$ which form a basis in $W$. We now use the Gram Schmidt orthogonalization process to othonormalize the first components $\alpha^i$ in the norm $\| \, \|_{2,\rho}$ as follows. Let

$$\beta^1 = \alpha^1/\|\alpha_1^1\|_{L_\rho^2(\Omega_p)} ,$$

$$B^2 = \{\alpha^2 - \langle\beta_1^1,\alpha_1^2\rangle_{L_\rho^2(\Omega_p)}\beta^1\}/\|\alpha_1^2 - \langle\beta_1^1,\alpha_1^2\rangle_{L_\rho^2(\Omega_p)}\beta^1\|_{L_\rho^2(\Omega_p)} ,$$

$$\vdots$$

$$\beta^n = \alpha^n - \sum_{i=1}^{n-1} \langle\beta_1^i,\alpha_1^n\rangle\beta^i/\|a_1^n - \sum_{i=1}^{n-1} \langle\beta_1^i,\alpha_1^n\rangle\beta_1^i\|_{L_\rho^2(\Omega_p)} .$$

For this scheme to be well defined, we need to show that $\|\alpha_1^n - \sum_{i=1}^{n-1} \langle\beta_1^i,\alpha_1^n\rangle\beta^i\|_{2,\rho} \neq 0$. Note that the space $S_n$ spanned by the $\beta^i$, $i = 1,\cdots,n$ is the same as that spanned by $\alpha^i$, $i = 1,\cdots,n$, because the matrix $A_n = [a_{ij}]$ that carries the column vector $\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}$ to $\begin{pmatrix}\beta_1\\ \vdots \\ \beta_n\end{pmatrix}$ is lower triangular.

$$A_n = \begin{pmatrix} a_{11} & & & & & & & & 0 \\ \cdot & \cdot & \cdot & & & & & & \\ \cdot & \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{nn} \end{pmatrix}$$

and its determinant is given by the product of the entries in the diagonal, a typical entry in the diagonal is

$$a_{ii} = [\|\alpha_1^i - \sum_{j=1}^{j=i-1} \langle\beta_1^j,\alpha_1^i\rangle\beta_1^j\|_{2,\rho}]^{-1} .$$

We now show that

$$\|\alpha_1^n - \sum_{i=1}^{n-1} \langle\beta_1^i,\alpha_1^i\rangle\beta_1^i\|_{2,\rho} \neq 0, \quad \forall n. \tag{6.10}$$

First note that $\beta^1 = \dfrac{\alpha^1}{\|\alpha_1^1\|_{2,\rho}}$ is well defined, since by assumption $0 < \|\alpha_1^1\| < \infty$. Now let $n$ be the first integer greater than one for which (6.10) fails, and

$$\alpha_1^n = \sum_{i=1}^{n-1} \langle\beta_1^i,\alpha_1^n\rangle_{2,\rho}\beta_1^i .$$

Then,

$$\alpha^n - \sum_{i=1}^{n-1} \langle\beta_1^i,\alpha_1^n\rangle_{2,\rho}\ \beta^i$$

$$= (\alpha_1^n,\alpha_2^n) - \sum_{i=1}^{n-1} \langle\beta_1^i,\alpha_1^n\rangle_{2,\rho}(\beta_1^i,\beta_2^i)$$

$$= (\alpha_1^n - \sum_{i=1}^{n-1} \langle\beta_1^i,\alpha_1^n\rangle_{2,\rho}\ \beta_1^i\alpha_2^n - \sum_{i=1}^{n-1} \langle\beta_1^i,\alpha_1^n\rangle\beta_2^i)$$

$$= (0,\ \alpha_2^n - \sum_{i=1}^{n-1} \langle\beta_1^i,\alpha_1^n\rangle_{2,\rho}\ \beta_2^i).$$

Since $\alpha^n \in W$ and $\sum_{i=1}^{n-1} \langle\beta_1^i,\alpha_1^n\rangle_{2,\rho}\ \beta^i \in W$, their difference,

$$(0,\ \alpha_2^n - \sum_{i=1}^{n-1} \langle\beta_1^i,\alpha_1^n\rangle_{2,\rho}\ \beta_2^i) \in W.$$

However, this is impossible unless

$$\alpha_2^n - \sum_{i=1}^{n-1} \langle\beta_1^i,\alpha_1^n\rangle_{2,\rho}\ \beta_2^i = 0.$$

For if $\xi = 0$, then $(\xi,A) \in W$ implies

$$n \times A = (-n \cdot \xi)B^v \quad \text{on} \quad \Gamma_p ,$$

i.e.

$$n \times A = 0 \quad \text{on} \quad \Gamma_p.$$

But as mentioned at the end of §4, the only solution on $\bar{V}_2(\Omega_v)$ of

$$\nabla \times \nabla \times A = 0,$$

$$n \times A = 0 \quad \text{on} \quad \Gamma_p \quad \text{and} \quad \Gamma_v ,$$

is

$$A \equiv 0.$$

Thus we have shown that

$$a_1^n = \sum_{i=1}^{n-1} <\beta_1^i,\alpha_1^n>_{2,\rho}\ \beta_1^i$$

implies

$$\alpha^n = \sum_{i=1}^{n-1} <\beta_1^i,\alpha_i^n>_{2,\rho}\ \beta^i = \sum_{i=1}^{n-1} (<\beta_1^i,\alpha_i^n>_{2,\rho} \sum_{j=1}^{n-1} a_{ji}\alpha^i)$$

$$= \sum_{i=1}^{n-1} (\sum_{i=1}^{n-1} <\beta_1^i,\alpha_1^n>_{2,\rho}\ a_{ji})\alpha^i = \sum_{i=1}^{n-1} c_i\alpha^i .$$

In other words, $\alpha^n,\alpha^{n-1},\cdots,\alpha^1$ are linearly dependent which is a contradiction.

Now since as mentioned above the space $S_n$ spanned by the first $n$ vectors $\alpha^n$ is the same as that spanned by the $\beta_n$, the closure in the $H$ norm of the space spanned by the $\beta_n$ is again $H$. Thus assertion (1) is proved.

(2) The closure of the set of all finite linear combinations of $\beta^n$ in the $W$ norm is identical to that of $\alpha_n$ and hence yields the space $W$.

In the following, we define the Galerkin approximations to the solution of the AEVP. Let

$$\Lambda^m(t) = (\xi^m(t),A^m(t)) = \sum_{i=1}^m c_i^m(t)\beta^i.$$

We shall call $\Lambda^m(t)$ a Galerkin approximation of the $m^{th}$ order to the AEVP if

$$\frac{d^2}{dt^2} < \sum_{i=1}^m c_i^m(t)\beta_1^i,\beta_1^j >_{2,\rho} + a((\sum_{i=1}^m c_i^m(t)\beta^i),\beta^j) = 0$$

$$\forall\ \beta^j \quad j = 1,\cdots,m, \tag{6.11}$$

and if $\Lambda^m(t)$ satisfies the initial conditions,

$$\Lambda^m(0) = (\xi^m(0),A^m(0)) = (\xi_0^m,A_0^m)$$

$$\frac{\partial\xi}{\partial t}(0) = \dot{\xi}_0^m .$$

Here by (6.5) and (6.6),

$$(\xi_0, A_0) = \sum_{i=1}^{\infty} c_i^0 (\beta_1^i, \beta_2^i) \qquad \text{in} \quad W,$$

$$\dot{\xi}_0 = \sum_{i=1}^{\infty} \tilde{c}_i^0 \, \beta_1^i \qquad \text{in} \quad P_{\Omega_p}(H),$$

so that

$$(\xi_0^m, A_0^m) = \sum_{i=1}^{m} c_i^0 (\beta_1^i, \beta_2^i); \quad \dot{\xi}_0^m = \sum_{i=1}^{m} \tilde{c}_i \beta_1^i \, .$$

Also (6.11) may be written as

$$\frac{d^2}{dt^2} \begin{pmatrix} c_1^m(t) \\ \vdots \\ c_m^m(t) \end{pmatrix} + B_m \begin{pmatrix} c_1^m(t) \\ \vdots \\ c_m^m(t) \end{pmatrix} = 0 \, , \tag{6.13}$$

where $B_m = [a(\beta^i, \beta^j)]$ is an $m \times m$ matrix. Note that the coefficient of $\frac{d^2}{d^2t} C_m(t)$ is the identity matrix because of the orthonormality of the $\beta_1^i$ in $L^2(\Omega_p)$.

The theory of ordinary differential equations now ensures a unique solution to (6.13) subject to the initial conditions (6.12) on the interval $[0,T]$. We now make use of the equations (6.1) to obtain a priori estimates on $\|\dot{\xi}^m(t)\|_{2,\rho}$ and $\|\Lambda^m(t)\|_W$. Multiply each of the equations (6.11) by $\dot{c}_i^m(t)$ and summing over $i$, we obtain:

$$\langle \ddot{\xi}^m(t), \dot{\xi}^m(t) \rangle_{2,\rho} + a(\Lambda^m(t), \dot{\Lambda}^m(t)) = 0. \tag{6.14}$$

Now because of the symmetry of $a((\xi, A), (\tilde{\xi}, \tilde{A})$ on $W$, the relationship may be written

$$\tfrac{1}{2} \frac{d}{dt} \{ \|\dot{\xi}^m(t)\|_{2,\rho}^2 + a(\Lambda^m(t), \Lambda^m(t)) \} = 0. \tag{6.15}$$

Thus after integration we obtain

$$\|\dot{\xi}^m(t)\|_{2,\rho}^2 + a(\Lambda^m(t), \Lambda^m(t)) = \|\dot{\xi}^m(0)\|_{2,\rho}^2 + a(\Lambda^m(0), \Lambda^m(0)). \tag{6.16}$$

Since the bilinear form $a((\xi, A), (\tilde{\xi}, \tilde{A}))$ is continuous on $W$, there exists $c_1 > 0$ such that

$$|a(\Lambda^m(0), \Lambda^m(0))| \leq c_1 \|\Lambda^m(0)\|_W^2.$$

Also by virtue of Bessel's inequality

$$\|\dot{\xi}^m(0)\|^2_{2,\rho} \le \|\dot{\xi}(0)\|^2_{2,\rho} = c_2.$$

Hence

$$\|\dot{\xi}^m(t)\|^2_{2,\rho} + a(\Lambda^m(t),\Lambda^m(t)) \le c_2 + c_1\|\Lambda^m(0)\|^2_W . \tag{6.17}$$

Furthermore, $\Lambda^m(0)$ converges to $\Lambda(0)$ in the $W$ norm, so there exists $c_3 > 0$ such that

$$\|\Lambda^m(0)\|_W \le c_3, \qquad \forall m .$$

Hence,

$$\|\dot{\xi}^m(t)\|^2_{2,\rho} + a(\Lambda^m(t),\Lambda^m(t)) \le c_2 + c_1c_3 = b_1. \tag{6.18}$$

Now because $a((\xi,A)(\xi,A))$ is coericive with respect to the $W$ norm, we have

$$a((\xi,A),(\xi,A)) + \lambda\|\xi\|^2_{2,\rho} \ge \delta\|(\xi,A)\|^2_W ,$$

$$(\xi,A) \in W.$$

Thus adding $\forall\lambda\|\xi\|^2_{2,\rho}$ to both sides of (6.18) and using the coerciveness yield

$$\|\dot{\xi}^m(t)\|^2_{2,\rho} + \delta\|(\xi^m,A^m)\|^2_W \le b_1 + \lambda\|\xi^m\|^2_{2,\rho} . \tag{6.19}$$

It follows that

$$\|\dot{\xi}^m(t)\|^2 \le b_1 + \lambda\|\xi^m\|^2_{2,\rho} . \tag{6.20}$$

Multiplying both sides of this inequality by $\|\xi^m(t)\|^2_{L^2_\rho(\Omega_p)}$ and using

$$\frac{d}{dt}\{\langle\xi^m(t),\xi^m(t)\rangle_{2,\rho}\} = 2\langle\xi^m(t), \frac{d}{dt}\xi^m(t)\rangle_{2,\rho}$$

$$\le 2\|\xi^m(t)\|_{2,\rho}\}\|\frac{d}{dt}\xi^m(t)\|_{2,\rho} ,$$

we obtain

$$\{ \tfrac{1}{2}\frac{d}{dt}\|\xi^m(t)\|^2_{2,\rho}\}^2 \le b_1\|\xi^m(t)\|^2_{2,\rho} + \lambda\|\xi^m(t)\|^4.$$

Therefore,

$$\frac{d}{dt}\|\xi^m(t)\|^2_{L^2_\rho(\Omega_p)} \le 2\{b_1\|\xi^m(t)\|^2_{2,\rho} + \lambda\|\xi^m(t)\|^4_{2,\rho}\}.$$

Since

$$\lambda\|\xi^m(t)\|^4_{2,\rho} + b_1\|\xi^m(t)\|^2_{2,\rho} \leq (\sqrt{\lambda}\|\xi^m(t)\|^2_{2,\rho} + \frac{b_1}{2\sqrt{\lambda}})^2$$

$$\leq (\sqrt{\lambda}\ \|\xi^m(t)\|^2 + \frac{b_1}{\sqrt{\lambda}})^2,$$

we obtain

$$\frac{d}{dt}\|\xi^m(t)\|^2_{2,\rho} \leq 2(\sqrt{\lambda}\ \|\xi^m(t)\|^2_{2,\rho} + \frac{b_1}{\sqrt{\lambda}}) \ . \tag{6.21}$$

Therefore,

$$\|\xi^m(t)\|_{L^2_\rho(\Omega_p)} \leq (\|\xi^m(0)\|^2_{2,\rho} + \frac{b_1}{\lambda}\ e^{2\sqrt{\lambda t}} - \frac{b_1}{\lambda} \ . \tag{6.22}$$

Let

$$b_2^m = \|\xi^m(0)\|^2_{2,\rho} \ ,$$

$$b_2 = \sup\{b_2^m\} \ .$$

Note that $b_2 < \infty$ is ensured from the inequality

$$\|\xi^m(0)\|^2_{2,\rho} \leq \|(\xi^m(0)),A^m(0)\|^2_W = \|\Lambda^m(0)\|^2_W \ .$$

Thus given $\varepsilon > 0$, there exists $M(\varepsilon)$ such that if $m > M(\varepsilon)$,

$$\|\Lambda^m(0)\|^2_W \leq \|\Lambda(0)\|^2_W + \varepsilon = c_3 + \varepsilon \ ,$$

because $\Lambda^m(0) \to \Lambda(0)$ in the $W$ norm.

Hence, for $m > M(\varepsilon)$

$$\|\xi^m(0)\|^2_{2,\rho} \leq c_3 + \varepsilon,$$

and in particular,

$$c_4 = \lim\sup\|\xi^m(0)\|^2 \leq c_3.$$

It follows from (6.22) that

$$\lim\sup\|\xi^m(t)\|^2_{2,\rho} \leq (c_4 + \frac{b_1}{\lambda})e^{2\sqrt{\lambda}t} - \frac{b_1}{\lambda} \quad \text{on} \quad [0,T]. \tag{6.23}$$

Returning now to equation (6.19), we may use (6.23) to derive similar upper bounds for $\|\dot{\xi}^m(t)\|^2_{L^2_\rho(\Omega_p)}$ and $\|(\xi^m,A^m)(t)\|^2_W$. It follows that

$$\lim\sup\{\|\dot{\xi}^m(t)\|^2_{2,\rho} + \delta\|(\xi^m,A^m)(t)\|^2_W\} \quad \leq \lim\sup\{b_1 + \lambda\|\xi^m\|^2_{2,\rho}\}$$

$$\leq b_1 + \lambda(\|\xi^m(0)\|^2_{2,\rho} + \frac{b_1}{\lambda})e^{2\sqrt{\lambda}t} - b_1 = \lambda(\|\xi^m(0)\|^2_{2,\rho} + \frac{b_1}{\lambda})e^{2\sqrt{\lambda}t}. \tag{6.24}$$

From (6.13) and (6.24) together with $e^{2\sqrt{\lambda}t} \leq e^{2\sqrt{\lambda}t}$ on $[0,T]$, we conclude that there exists $M > 0$ such that on $[0,T]$

$$\|\xi^m(t)\|^2_{2,\rho} \leq M, \qquad \forall m ,$$

$$\|\dot{\xi}^m(t)\|^2_{2,\rho} \leq M, \qquad \forall m ,$$

$$\|(\xi^m, A^m)\|^2_W \leq M, \qquad \forall m .$$

Thus,

$$\{\Lambda^m(t)\} = ((\xi^m(t), A^m(t)))\}$$

forms a bounded sequence in the Banach space

$$L^\infty([0,T]:W) \subset L^2([0,T]:W);$$

$\{\dot{\xi}^m(t)\}$ forms a bounded sequence on the Banach space

$$L^\infty([0,T]:W) \subset L^2([0,T]:H).$$

Therefore we may extract a subsequence such that

$$\begin{aligned} &\Lambda^{n_m}(t) \to \Lambda(t) \quad \text{in} \quad L^\infty([0,T]:W), \\ &\Lambda^{n_m}(t) \to \Lambda(t) \quad \text{in} \quad L^2([0,T]:W), \\ &\dot{\xi}^{n_m}(t) \to \tilde{\xi}(t) \quad \text{in} \quad L^\infty([0,T)]:P_{\Omega_p}(H)). \end{aligned} \tag{6.26}$$

In particular from (6.26) it follows that

$$\xi^{n_m}(t) \to \xi(t) \quad \text{in} \quad L^2([0,T]:L^2_\rho(\Omega_p)).$$

Thus $\tilde{\xi}(t) = \dot{\xi}(t)$, because of the uniqueness of the representation of a vector-valued distributional derivative (Lions and Magenes, 1972) as an element of

$$(L^\infty([0,T]:L^2_\rho(\Omega_p)) \subset L^1([0,T]:L^2_\rho(\Omega_p)).$$

We next show that the function $\Lambda(t)$ defined in this manner is indeed a solution of the AEVP. For this, let $\psi \in C_0[0,T]$ be such that $\psi(0) = \psi(T) = 0$. Let $\psi^i(t) = \psi(t)\beta^i$, $\psi^i_1(t) = \psi(t)\beta^i_1$. We then multiply

(6.11) for $\Lambda^{n_m}$ by $\psi(t)$ and then integrate by parts with respect to $t$ over the interval $[0,T]$ to obtain

$$\int_0^T \langle \dot{\xi}^{n_m}(t), \dot{\psi}_1^i(t)\rangle_{2,\rho} + a(\Lambda^{n_m}(t), \psi^i(t))dt = 0. \tag{6.27}$$

Taking the limit as $n_m \to \infty$ and using the weak convergence of $\Lambda^{n_m}(t)$ to $\Lambda(t)$ in $L^2([0,T]:W)$, we obtain

$$\begin{aligned} 0 &= \int_0^t \langle \dot{\xi}(t), \dot{\psi}_1^i(t)\rangle_{2,\rho} + a(\Lambda(t), \psi^i(t))dt \\ &= \int_0^T \{\langle \dot{\xi}(t), \beta_1^i\rangle_{2,\rho}\, \dot{\psi}(t) + a(\Lambda(t), \beta^i)\psi(t)\}dt \\ &= \int_0^T \{\frac{d}{dt} \langle \dot{\xi}(t), \beta_1^i\rangle_{2,\rho}\, \dot{\psi}(t) + a(\Lambda(t), \beta^i)\psi(t)\}dt. \end{aligned}$$

Thus

$$\int_0^T \frac{d}{dt} \langle \dot{\xi}(t), \beta_1^i\rangle_{2,\rho}\, \dot{\psi}(t)dt = \int_0^t a(\Lambda(t), \beta^i)\psi(t)dt$$

for all $\psi(t) \in D[0,T)$.

We show $g(t) = a(\Lambda(t), \beta^i) \in L^2[0,T]$. This is immediate if one uses the continuity of $a(\cdot,\cdot)$ in the $W$ norm, i.e. there exist $c > 0$ such that

$$|a(\Lambda(t), \beta^i)| \le C\|\Lambda(t)\|_W \|\beta^i\|_W ,$$

and if we pass to the limit in (6.25), we obtain

$$\|\Lambda(t)\|_W \le M.$$

Therefore,

$$|a(\Lambda(t), \beta^i)| \le CM\|\beta^i\|_W, \tag{6.28}$$

so that

$$g(t) \in L^\infty[0,T] \subset L^2[0,T].$$

Hence from (6.28) we may deduce

$$\frac{d^2}{dt^2} \langle \xi(t), \beta_1^i\rangle_{2,\rho} \in L^2[0,T]. \tag{6.29}$$

The derivative is again taken in the sense of distributions. Furthermore,

$$\frac{d^2}{dt^2} \langle \xi(t), \beta_1^i\rangle_{2,\rho} = -a(\Lambda(t), \beta^i) .$$

Now since this equation holds for arbitrary $\beta^i$ and since the $\beta^i$ form a complete set in $W$, it holds for every $(\tilde{\xi},\tilde{A}) \in W$. We conclude that

$$\frac{d^2}{dt^2} \langle \xi(t),\tilde{\xi} \rangle_{2,\rho} + a(\Lambda(t),(\tilde{\xi},\tilde{A})) = 0 \ , \qquad (6.30)$$

$$\forall (\tilde{\xi},\tilde{A}) \in W \ ,$$

which completes the demonstration of existence of a solution to the AEVP.

§6.2. Solution of the EVP

The solution to the EVP is constructed in the following way. To the solution of the AEVP defined by (6.5) to (6.9), we add a pair $(0,A')$ defined on $\Omega_p \times \Omega_v$ with $A'$ being a solution of

$$\begin{aligned} &\nabla \times \nabla \times A'(t) = 0 \ , \\ &\quad \nabla \cdot A'(t) = 0 \ , \qquad (6.31) \\ &n \times A' = 0 \quad \text{on} \quad \Gamma_p \cup \Gamma_v \ , \\ &\int_{\Gamma_v} (A' \cdot n) ds = \sigma(t) \quad \text{on} \quad [0,T], \end{aligned}$$

where $A'(t)$ is taken from $\bar{V}_3$ (see 4.4). That is,

$$\begin{aligned} &\nabla \times A'(t) = 0 \qquad n \times A'(t) = 0 \quad \text{on} \quad \Gamma_p \cup \Gamma_v \ , \\ &\nabla \cdot A'(t) = 0 \qquad \int_{\Gamma_v} A \cdot n \ ds = \sigma(t) \quad \text{on} \quad [0,T]. \end{aligned}$$

We note, in particular, that if we define $A'$ as the unique solution of

$$\begin{aligned} &\nabla \times A_1' = 0, \\ &\nabla \cdot A_1' = 0 \ , \\ &\int_{\Gamma_v} A_1' \cdot n \ ds = 1 \ , \end{aligned}$$

then, $A'(t) = \sigma(t)A_1'$, by linearity and uniqueness. The solution $(\xi,\hat{A})(t)$ of the EVP defined by

$$(\xi,\hat{A})(t) = (\xi,A)(t) + (0,A')(t) = (\xi,A + A')(t)$$

has the property

$$\|(\xi,\hat{A})\|_W = \|(\xi,A)\|_W \ .$$

This is shown as follows. Since the first term $\xi$ is unchanged, all terms entering into $\|(\xi,\hat{A})\|_W$ involving only $\xi$ are unchanged. The vacuum contribution is given by

$$\int_{\Omega_v} |\nabla \times (A + A')|^2 dv = \int_{\Omega_v} |\nabla \times A + \nabla \times A'|^2 dv$$

$$= \int_{\Omega_v} |\nabla \times A|^2 dv,$$

since

$$\nabla \times A'(t) \equiv 0.$$

Hence we have (6.1)

$$\frac{d^2}{dt^2} \langle \xi,\tilde{\xi}\rangle_{2,\rho} + a((\xi,\hat{A}),(\tilde{\xi},\hat{\tilde{A}})) = 0$$

$$\forall (\tilde{\xi},\hat{\tilde{A}}) \in W^+ ,$$

and also the natural boundary condition $L(\xi,\hat{A}) = 0$ holds whenever the integration by parts is permissible (§4). The initial conditions are clearly also satisfied because of (6.6) to (6.8).

In summary, the solution to the EVP was obtained in two stages. First an AEVP was solved, and then we added to the vacuum part of this solution a solution to the elliptic system (6.31). As noted above, this elliptic system with the given flux and initial conditions has a unique solution, therefore to show the uniqueness of the solution to the EVP it suffices to establish the uniqueness of the solution to the AEVP.

Let $\Lambda(t) = (\xi(t),A(t))$ be a solution to the AEVP subject to the initial conditions

$$\Lambda(0) = 0 \ , \quad \dot{\xi}(0) = 0.$$

Let $s \in (0,T)$. Let $(\tilde{\xi}(t),\tilde{A}(t))$ be defined by

$$\tilde{\xi}(t) = \begin{cases} -\int_t^s \xi(\sigma)d\sigma & t < s \\ 0 & t > s \ , \end{cases}$$

$$\tilde{A}(t) = \begin{cases} -\int_t^s A(\sigma)d\sigma & t < s \\ 0 & t > s\,. \end{cases}$$

Note that $\tilde{\Lambda}(t) = (\tilde{\xi}(t),\tilde{A}(t)) \in W \;\; \forall t \in [0,T]$, simply by interchanging integration with respect to time and differentiation in space. It follows that

$$\frac{\partial}{\partial t}\tilde{\xi}(t) = \xi(t), \quad \frac{\partial}{\partial t}\tilde{A}(t) = A(t)\,,$$

$$\tilde{\xi}(\hat{t}) = \tilde{A}(\hat{t}) = 0\,, \quad \hat{t} > s\,.$$

Moreover, note that $\frac{\partial^2}{\partial t^2}\xi(t)$ is an element of $W^*$, the dual of $W$. Let $<\frac{\partial^2}{\partial t^2}\xi(t),\tilde{\xi}(t)>_{W^*\times W}$ denote the inner product between $W$ and $W^*$ in the duality between $W$ and $W^*$.

Since $\Lambda(t)$ is a solution to AEVP with zero initial conditions, we have

$$<\frac{\partial^2}{\partial t^2}\xi,\tilde{\xi}>_{W\times W^*} + a(\Lambda(t),\tilde{\Lambda}(t)) = 0.$$

Hence,

$$\int_0^t\{<\frac{\partial^2}{\partial t^2}\xi,\tilde{\xi}>_{W\times W^*} + a(\Lambda(t),\tilde{\Lambda}(t))\}dt = 0.$$

Integrating by parts and using the initial condition $\dot{\xi}(0) = 0$ yields

$$\int_0^T\{-<\dot{\xi}(t),\xi(t)>_{2,\rho} + a(\Lambda(t),\tilde{\Lambda}(t))\}dt = 0.$$

Therefore, since $(\xi(t),A(t)) = (0,0)$ for $t > s$

$$\int_0^s\{-<\dot{\xi}(t),\xi(t)>_{2,\rho} + a(\Lambda(t),\tilde{\Lambda}(t))\}dt = 0,$$

and using $\frac{\partial}{\partial t}\tilde{\Lambda}(t) = \Lambda(t)$, we obtain

$$\int_0^s \{\frac{\partial}{\partial t}\,[\|\xi(t)\|^2_{2,\rho} + a(\tilde{\Lambda}(t),\tilde{\Lambda}(t))]\} = 0.$$

It follows that

$$a(\tilde{\Lambda}(0),\tilde{\Lambda}(0)) + \|\xi(s)\|^2 = 0.$$

Now adding $\lambda\|\tilde{\xi}(0)\|^2_{2,\rho}$ to both sides and using the coerciveness of $a(\cdot,\cdot)$ we obtain

$$\delta\|\Lambda(0)\|^2_W + \|\tilde{\xi}(s)\|^2_{2,\rho} \leq \lambda\|\tilde{\xi}(0)\|^2\,,$$

or

$$\|\Lambda(0)\|_W^2 + \|\xi(s)\|_{2,\rho}^2 \leq d\|\xi(0)\|^2,$$

where $d > 0$. We set $W = (W_1, W_2)$ with

$$W_1(t) = \int_0^t \tilde{\xi}(\sigma)d\sigma ,$$

$$W_2(t) = \int_0^t A(\sigma)d\sigma.$$

Thus,

$$\|W(s)\|_W^2 + \|\tilde{\xi}(s)\|_{2,\rho}^2 \leq d\|W_1(s)\|_{2,\rho}^2 ,$$

or

$$\|W(s)\|_W^2 + \|\tilde{\xi}(s)\|_{2,\rho}^2 \leq d\|\int_0^s \tilde{\xi}(\sigma)d\sigma\|_{2,\rho}^2 .$$

Adding the positive quantity $\int_0^s d\|W(\sigma)\|_W^2 d\sigma$ to the right hand side of this inequality we obtain

$$\frac{d}{ds}\,\phi(s) \leq d\phi(s), \tag{6.32}$$

where $\phi(s) = \int_0^s \{\|W(\sigma)\|_W^2 + \|\xi(\sigma))\|_{2,\rho}^2\}d\sigma$. Therefore, since $\phi(0) = 0$ , integration of (6.32) yields

$$\phi(s) \equiv 0 \quad \text{and} \quad \frac{d\phi}{ds} \equiv 0 \quad \text{on} \quad [0,T].$$

Consequently,

$$\|\xi(t)\|_{2,\rho} \equiv 0 \quad \text{implies} \quad \xi(t) \equiv 0 \quad \text{on} \quad [0,T].$$

It follows that $A(t) \equiv 0$ by Theorem 4.2. Hence, $\Lambda(t) = (\xi(t),A(t)) \equiv 0$. This establishes that the only solution to the AEVP is the null solution. The uniqueness result for the EVP then follows trivially.

In conclusion, we note that the solution to the EVP exists for all time $t > 0$. This result follows trivially from the global estimates in §6. Following a theorem of Lions and Magenes (1972), we may summarize our results as

<u>Theorem 6.1.</u> There exists a unique solution to the EVP $(\xi,A)(t)$ in $C([0,\infty):W^+)$ with $\dot{\xi}(t)$ in $C([0,\infty):P_{\Omega_0}(H^+))$ such that

$$\frac{d^2}{dt^2} \langle \xi(t), \tilde{\xi} \rangle_{2,\rho} + a((\xi, A)(t), (\tilde{\xi}, \tilde{A})) = 0 \qquad \forall\ (\tilde{\xi}, \tilde{A}) \in W^+,$$

subject to

$$(\xi(0), A(0)) = (\xi_0, \tilde{A}_0) \in W^+,$$

$$\frac{\partial \xi(0)}{\partial t} = \dot{\xi}_0 \in P_{\Omega_p}(H^+),$$

and the flux condition

$$\int_{\Gamma_v} \hat{A}(t) \cdot n \, dS = \sigma(t).$$

where $\sigma(t)$ is a prescribed continuous function of $t$.

§7. Discussion

First we make some remarks about the solutions to the EVP. In the process of establishing existence of a solution to the AEVP, we construct Galerkin approximations and extracted a subsequence of the approximations which converges to the solution weakly. We note that the entire Galerkin sequence is in fact weakly convergent to the solution. The proof is based upon a well-known compactness property (Yosida, 1978) which we formulate as the following:

<u>Lemma:</u>

In a Banach space $B$, the first two assertions imply the third and conversely

(1) $u_n \in B$, $u_n$ weakly compact,

(2) all weakly convergent subsequences of $u_n$ must converge weakly to the same limit, denoted $u$, imply that

(3) $u_n$ is weakly convergent.

We make use of this lemma in the following way:

Recall that $\Lambda_n$ was weakly compact in $L^{\infty}([0,T]:W)$ and in $L^2([0,T]:W)$ (see §6). Also $\dot{\xi}^n$ was weakly compact in $L^{\infty}([0,T]:H)$ and in $L^2([0,T]:H)$. Consider the space

$$B = L^2(0,T]:W) \times L^2([0,T]:H).$$

Let $u_n = (\Lambda_n, \dot{\xi}_n)$. $u_n$ is weakly compact in $B$, which is a Hilbert space with norm

$$\|(\Lambda_n, \dot{\xi}_n)\| = \{\|\Lambda_n\|^2_{L^2([0,T]:W)} + \|\dot{\xi}_n\|^2_{L^2([0,T]:W)}\}.$$

Furthermore, if $u_{n_m}$ is a subsequence which converges weakly in $B$, then exactly the same sequence of arguments as given for (6.27) to (6.30) shows that the limit of $u_{n_m}$ is indeed a solution to the auxiliary evolutionary

variational problem. This shows that the entire Galerkin approximation is weakly convergent to the solution.

In order to be sure that our solution of (EVP) is a "good" solution of (MIBVP) we should verify that it satisfies the following two criteria:

(1) Any classical solution of MIBVP, if it exists, is a solution of EVP.

(2) Any classical solution of EVP is a classical solution of MIBVP.

To verify (1) we note that the sequence of steps leading from (3.1) to (3.10) performed now for pairs $(\tilde{\xi},\tilde{\tilde{A}}) \in W^+$ shows that

$$\langle\rho\ddot{\xi},\xi\rangle_{2,\rho} + a((\xi,A),(\tilde{\xi},\tilde{\tilde{A}})) = \int_{\Gamma_p} (\tilde{\xi} \cdot n)L(\xi,A)ds,$$

if $(\xi,A)(t)$ is a classical solution of MIBVP. The right hand side however vanishes, since the boundary condition (3.4) is satisfied by a solution to MIBVP, if it exists.

To verify (2) we start from

$$\frac{\partial^2}{\partial t^2}\langle\xi,\tilde{\xi}\rangle_{2,\rho} + a((\xi,A),(\tilde{\xi},\tilde{A})) = 0,$$

$$(\tilde{\xi},\tilde{\tilde{A}}) \in W^+,$$

and assume $(\xi,A)(t)$ is a $C^2$ function on $\bar{\Omega}_p \times \bar{\Omega}_v$, and $C^2$ in time.

We then may integrate by parts the various integrals in $a((\xi,A),(\tilde{\xi},\tilde{A}))$, and bring the operator $\frac{d^2}{dt^2}$ inside the scalar product to get

$$\frac{d^2}{dt^2}\langle\xi(t),\tilde{\xi}\rangle_{2,\rho} + a((\xi,A),(\tilde{\xi},\tilde{\tilde{A}})) =$$

$$= \langle\frac{\partial^2}{\partial t^2}\xi(t),\tilde{\xi}\rangle_{2,\rho} - \langle F(\xi),\tilde{\xi}\rangle_{2,\rho} + \langle\nabla \times \nabla \times \tilde{A},A\rangle_{2,v}$$

$$= \int_{\Gamma_p} (\tilde{\xi} \cdot n)L(\xi,A)ds,$$

where $A \in W^+$, $\nabla \times \nabla \times A = 0$.

Furthermore since $C_0^\infty(\Omega_p)$ $P_{\Omega_p}(W^+)$, we may choose for $\tilde{\xi}$, any $C_0^\infty(\Omega_p)$ function, so that the boundary contribution vanishes, therefore

$$\frac{\partial^2}{\partial t^2}\ \xi(t) = F(\xi(t)) \quad \text{in} \quad D'([0,T]:C^2(\Omega_p)),$$

(Brezis, 1974), since $F(\xi(t)) \in C^0[0,T]:C^2(\Omega_p)]$,

$$\xi(t) \in C^2([0,T]:C^2(\Omega_p)).$$

Finally, since $H^1(\Omega_p)^3$ $P_{\Omega_p}(W^+)$, we may take for $(\tilde{\xi} \cdot n)$ any element in $L^2(\Gamma_p)$ so that

$$L(\xi,A) = 0 \quad \text{in} \quad L^2(\Gamma_p) = (L^2(\Gamma_p))^*.$$

Thus, since $L(\xi,A)$ is represented by an element in $C^0(\Gamma_p)$,

$$L(\xi,A) = 0 \quad \text{in} \quad C^0(\Gamma_p).$$


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## REFERENCES


Bernstein, I., Freeman, E. A., Kruskal, M. D., and Kulsrud, R. M., An energy principle for hydromagnetic stability problems. Proc. Roy Soc. London Ser A 244, (1958), 17-40.

Blank, A. A., Grad, H., Friedrichs, K. O., Theory of Maxwell's Equations Without Displacement Current. N.Y.U. Report N.Y.O.-6486-v, Courant Institute of Math. Sciences 1957.

Friedrichs, K. O., Differential Forms on Riemannian Manifolds, Comm. Pure and Appl. Math. 8, (1955), 551-590.

Furth, H. P., Process Towards a Tokomak Fusion Reactor, Scientific American, 241, (1979), August 51-61.

Goedbloed, J. P., Sakanaka, P. H., A New Approach to MHD Stability, I, II, N.Y.U. Reports c00-3077-29, c00-3077-30, Courant Institute of Math. Sciences, 1973.

Grad, H., Plasmas, Physics Today, 22, (1969), December 34-44.

Green, J. M., Johnson, L., Interchange Instabilities in Ideal Hydromagnetic Theory, Plasma Physics, 4, (1968), 729-741.

Grossman, W., Analysis of High Beta Plasmas, Lecture Notes, Max-Plank Institute for Plasma Physics, 4, 1968.

Kress, R., Die Behandlung zweier Randwertprobleme für Vektorielle Poissongleichung nach einer Integraogleichungsmethode, Arch. Rat. Mech. Anal. 39, (1970), 206-226.

Laval, G., Mercier, C., and Pellat, R., Necessity of the Energy Principle for Magnetostatic Stability. Nuclear Fusion 5, (1965), 156-158.

Lions, J. L. and Magenes, E., Non Homogeneous Boundary Value Problems, Vol. 1, Springer-Verlag, 1972.

Lüst, V. R., Martensen, E., Zur mehrwertigkeit des Skalaren Magnetischen Potentials beim hydromagnetischen Stabilitatsproblem eines Plasmas, Z. Naturforschung 15 (1960), 706-713.

Rodiquez, G., UCLA E.E.E. Thesis, 1974.

Spiess, G. O., Elements of MHD Stability Theory, NYO Report, c00-3077-137, Courant Institute of Math. Sciences, 1976.

Temam, Roger, Navier-Stokes Equations: Theory and Numerical Analysis, North Holland, 1977.

Werner, P., Self-adjoint extension of a Laplace operator with respect to electrical and magnetic boundary conditions, J. Math. Anal. Appl., 20, (1979), 131-160.

Yosida, K., Functional Analysis. Springer-Verlag, 1965.

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## APPENDIX

We now establish the identity

$$\nabla(\xi \cdot \nabla \times B \times B) = \nabla(\xi \cdot \nabla B) \times B + \nabla \times B \times (\xi \cdot \nabla B)$$
$$- [((\nabla\xi)^T \cdot \nabla) \times B] \times B + (\nabla \times B \times B) \cdot \nabla\xi. \qquad \text{(A.1)}$$

This identity need not be true for all pairs $\xi$ and $B$, and only for those where $B$ satisfies:

$$\nabla \cdot B = 0$$
$$\nabla \times B \times B = \nabla p \qquad \text{(A.2)}$$

and we assume that

$$(\nabla \times B \times B)^i_j = (\nabla \times B \times B)^j_i \qquad \text{(A.3)}$$

which would follow if $p$ is twice continuously differentiable (equality of the mixed second partial derivatives). To prove the identity we first establish its validity for $\xi = e_1$, $\xi = e_2$, or $\xi = e_3$ the Cartesian basis vectors and $B$ satisfying (A.1).

For $\xi = e_i$, the identity

$$\nabla(\xi \cdot \nabla \times B \times B) = \nabla \times (\xi \cdot \nabla B) \times B + \nabla \times B \times (\xi \cdot \nabla B)$$
$$- ((\nabla\xi)^t \cdot \nabla) \times B \times B + \nabla \times B \times B,$$

becomes

$$\nabla(\nabla \times B \times B)^i = \nabla \times (\frac{\partial}{\partial x_i} B) \times B + \nabla \times B \times (\frac{\partial}{\partial x_i} B),$$

since

$$\xi \cdot \nabla = \frac{\partial}{\partial x_i} .$$

Now the R. H. S. can be written

$$\frac{\partial}{\partial x_i} (\nabla \times B \times B),$$

so the identity (A.1) reduces to (A.3). Now if the identity is true for $\xi_1$ and $\xi_2$ it is clearly true for $\xi = \xi_1 + \xi_2$ since all terms split into a sum of terms involving $\xi_1$ and $\xi_2$, it is slightly harder to show that if it is true for $\xi$ it is true for $\alpha(x)$ where $\alpha(x)$ is a function of

$x_1, x_2, x_3$, i.e. we seek to show:

$$\nabla(\alpha\xi \cdot \nabla \times B \times B) = \nabla \times ((\alpha\xi) \cdot \nabla B) \times B$$

$$+ \nabla \times B \times ((\alpha\xi) \cdot \nabla B) - (\nabla(\alpha\xi)^T \cdot \nabla) \times B \times B$$

$$+ \nabla \times B \times B \cdot \nabla(\alpha\xi), \tag{A.4}$$

follows from

$$\nabla(\xi \cdot \nabla \times B \times B) = \nabla \times (\xi \cdot \nabla B) \times B + \nabla \times B \times (\xi \cdot \nabla B)$$

$$- ((\nabla\xi)^T \cdot \nabla) \times B \times B + \nabla \times B \times B \cdot \nabla\xi. \tag{A.5}$$

For this note that using

$$\nabla(fg) = (\nabla f)g + (\nabla g)f, \tag{A.6}$$

the L. H. S. of (A.4) is

$$\alpha\nabla(\xi \cdot \nabla \times B \times B) + (\nabla\alpha)((\nabla \times B \times B) \cdot \xi).$$

Using (A.6) and the identity

$$\nabla \times (fv) = f\nabla \times v + \nabla f \times v ,$$

the R. H. S. becomes

$$\alpha(\nabla \times (\xi \cdot \nabla B) \times B) + \{\nabla\alpha \times (\xi \cdot \nabla B)\} \times B$$

$$+ \alpha((\nabla \times B) \times (\xi \cdot \nabla)B \times \xi - \alpha[(\nabla\xi)^T \cdot \nabla] \times B \times B$$

$$- [ \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix} (\xi_1, \xi_2, \xi_3) \cdot \nabla] \times B \times B + \alpha\nabla \times B \times B \cdot \nabla\xi$$

$$+ (\nabla \times B \times B \cdot \xi)\nabla\alpha,$$

where

$$\alpha_x = \frac{\partial\alpha}{\partial x},$$

$$\alpha_y = \frac{\partial\alpha}{\partial y},$$

$$\alpha_z = \frac{\partial\alpha}{\partial z}.$$

Cancelling $(\nabla\alpha)\ (\nabla \times B \times B \cdot \xi)$ from both sides and using (A.5), it remains to show

$$\{\nabla\alpha \times (\xi \cdot \nabla B)\} \times B = [\{ \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix} (\xi_1, \xi_2 \xi_3)\} \cdot \nabla] \times B \times B.$$

In fact,

$$\nabla\alpha \times (\xi \cdot \nabla B) = \{ \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix} (\xi_1, \xi_2, \xi_3)\} \cdot \nabla] \times B.$$

The L. H. S. is

$$\det\begin{pmatrix} i & j & k \\ \alpha x & \alpha y & \alpha z \\ \sum_{i=1}^{i=3} \xi^i \frac{\partial}{\partial x_i} B^1 & \sum_{i=1}^{i=3} \xi^i \frac{\partial}{\partial x_i} B^2 & \sum_{i=1}^{i=3} \xi^i \frac{\partial}{\partial x_i} B^3 \end{pmatrix}$$

and the R. H. S. is:

$$\det\begin{pmatrix} i & j & z \\ \sum_{i=1}^{3} \alpha x \, \xi^i \frac{\partial}{\partial x_i} & \sum_{i=1}^{3} \alpha y \, \xi^i \frac{\partial}{\partial x_i} & \sum_{i=1}^{3} \alpha z \, \xi^i \frac{\partial}{\partial x_i} \\ B^1 & B^2 & B^3 \end{pmatrix}$$

which are clearly equal.

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The linear MHD stability of a confined plasma is generally studied by means of energy principles. Up to date, these energy principles have never been justified rigorously, and the existence of a solution to the linearized equations is also tacitly assumed. In this report, based up a variational

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approach we shall first establish the existence and uniqueness of a generalized solution to the linearized Lundquist equations for a toroidal plasma confined in a conducting shell. In a subsequent report, the so-called modified energy principle, which includes linear energy principle as a special case, will be justified rigorously and a solid foundation is then laid for the application of these energy principles to the linear MHD stability of a confined plasma.

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